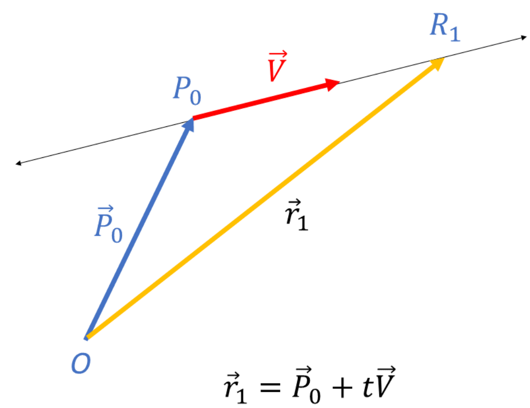
V6 Vector Equation of a Line



Introduction

You have probably been taught that a line in the $x - y$ plane can be represented in the form

$$y = mx + c$$

where m is the gradient (or slope) of the line and c is the $y-$intercept. But how do we define a line in three dimensions (3D)? [1]

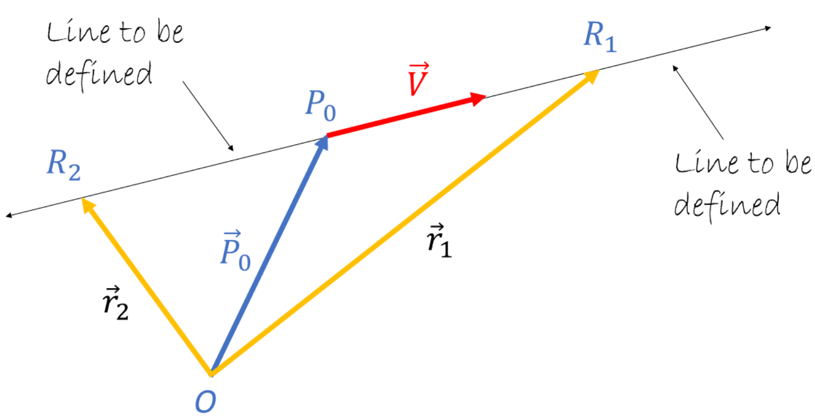
This module deals with the equation of a line in 3D.

[1] Three dimensional space (3D) is a geometric setting in which three co-ordinates are required to determine the position of a point. This is different to when we are dealing with points in the $x - y$ plane where only two co-ordinates are required.

Vector Equation of a Line in 3D

A line in 3D space can be described by a point and a vector along the line. There is only one line that can be drawn through a given point in the direction of a given vector.

Consider the figure below:



Here

- $\vec{r}_1$ and $\vec{r}_2$ are two points on the line;
- $P_0\,(x, y, z)$ is a point on the line and $\vec{P}_0$ is it's position vector;
- $\vec{V}$ is a vector along the line;

- t is a constant. That is $t \in \mathbb{R}$.

To get to the point R_1 you first need to get onto the line. You do this by traveling along $\vec{P_0}$ to the point P_0 on the line and then traveling a distance along the line in the direction of vector $\vec{V}$. So we could write

$$\vec{r}_1 = \vec{P_0} + t\vec{V}$$

where t is a positive constant. The quantity $t\vec{V}$ represents the distance and the direction we need to go along the line to get to the point R_1 from P_0 .

To get to the point R_2 you first need to get onto the line. You do this by traveling along $\vec{P_0}$ to the point P_0 on the line and then traveling a distance along the line in the **opposite** direction of vector $\vec{V}$. So we could write

$$\vec{r}_1 = \vec{P_0} + t\vec{V}$$

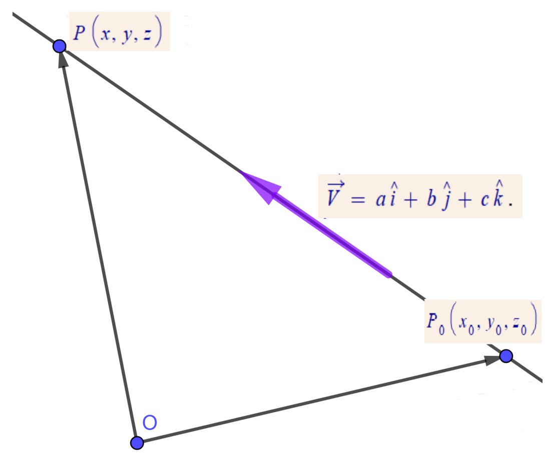
where t is a **negative** constant (because we are going in the direction opposite to $\vec{V}$).

Considering the above, we hope you understand the equation of a line, in vector form, is

$$\vec{r} = \vec{P_0} + t\vec{V} \tag{1}$$

where $t \in \mathbb{R}$ is a constant.

It is also possible to define the line in other forms as described below.



Parametric Equation of a Line in 3D

Suppose we wish to find the line that goes through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\overrightarrow{V} = a\hat{i} + b\hat{j} + c\hat{k}$. Consider a general point $P(x, y, z)$ that lies on the line that goes through the point $P_0(x_0, y_0, z_0)$ as shown at right. We can define a vector $\overrightarrow{P_0P}$ by:

$$\overrightarrow{P_0P} = (x - x_0)\,\hat{i} + (y - y_0)\,\hat{j} + (z - z_0)\,\hat{k}. \tag{2}$$

This vector will obviously be in the same direction as $\overrightarrow{V}$ but will, in all likelihood have a different magnitude. In other words $\overrightarrow{P_0P}$ will be a multiple of the direction vector $\overrightarrow{V}$. So $\overrightarrow{P_0P} = t\overrightarrow{V}$ where $t \in \mathbb{R}$ is some number.

Hence:

$$\begin{aligned}\overrightarrow{P_0P} &= t\overrightarrow{V} \\ &= t\left(a\hat{i} + b\hat{j} + c\hat{k}\right) \\ &= at\hat{i} + bt\hat{j} + ct\hat{k}. \quad (3)\end{aligned}$$

Equating components of (2) and (3) we have,

$$\begin{aligned}(x - x_0) &= at \\ x &= x_0 + at \qquad (4)\end{aligned}$$

and

$$\begin{aligned}(y - y_0) &= bt \\ y &= y_0 + bt \qquad (5)\end{aligned}$$

and

$$\begin{aligned}(z - z_0) &= ct \\ z &= z_0 + ct. \qquad (6)\end{aligned}$$

These equations $x = x_0 + at$, $y = y_0 + bt$ and $z = z_0 + ct$ are called the parametric equations of the line that contains the point (x_0, y_0, z_0) and has the direction vector $\overrightarrow{V} = a\hat{i} + b\hat{j} + c\hat{k}$. The variable $t \in \mathbb{R}$ is called a parameter.

Symmetric Form of the Line

By rearranging (4) to (6) above we find:

$$\begin{aligned}t &= \frac{x - x_0}{a} \\ t &= \frac{y - y_0}{b} \\ t &= \frac{z - z_0}{c}.\end{aligned}$$

Since they are all equal, we can say that:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad (7)$$

This is called the symmetric form of the equation of the line.

Summary

A line in 3D may be represented:

1. In vector form (see (1) above)
2. In parametric form (see $(4-6)$ above)
3. In symmetric form (see (7)above).

Example 1

Find the parametric and symmetric equations of the line that goes through the point $(3, 2, 3)$ and is in the direction of the vector $2\hat{i} + \hat{j} - 5\hat{k}$.

Solution:

Using equations $(4-6)$ the parametric equations will be :

$$\begin{aligned} x &= 3 + 2t \\ y &= 2 + t \\ z &= 3 - 5t. \end{aligned}$$

Using equation (7) the symmetric equations will be:

$$\frac{x-3}{2} = \frac{y-2}{1} = \frac{z-3}{-5}$$

or

$$\frac{x-3}{2} = y - 2 = \frac{3-z}{5}.$$

Example 2

Find the parametric and symmetric equations of the line that goes through the point $(1, 3, 2)$ in the direction of the vector $\overrightarrow{V} = \hat{j} - 2\hat{k}$.

Solution:

The parametric equations will be:

$$\begin{aligned} x &= 1 + 0t \\ &= 1 \\ y &= 3 + t \\ z &= 2 - 2t. \end{aligned}$$

The symmetric equation will be:

$$x = 1;\; y - 3 = \frac{z-2}{-2}$$

or

$$x = 1; \, y - 3 = -\frac{1}{2}z + 1.$$

Example 3

Find the direction vector of the line :

$$\frac{x-6}{2} = \frac{y-2}{-1} = \frac{z}{3}.$$

Solution:

The line is in symmetric form and so

$$\begin{aligned} \frac{x-6}{2} &= t \\ \frac{y-2}{-1} &= t \\ \frac{z}{3} &= t \end{aligned}$$

for any $t \in \mathbb{R}$. Rearranging gives the parametric form:

$$\begin{aligned} x &= 6 + 2t \\ y &= 2 - t \\ z &= 3t. \end{aligned}$$

This may be written in vector form as

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \left(6\hat{\imath} + 2\hat{\jmath} + 0\hat{k}\right) + t\left(2\hat{\imath} - \hat{\jmath} + 3\hat{k}\right)$$

which is a line through the point $(6, 2, 0)$ in the direction $\left(2\hat{\imath} - \hat{\jmath} + 3\hat{k}\right)$.

Hence the direction vector is $\overrightarrow{V} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$. Note that the direction vector of the line could be any multiple of this vector.

Example 4

A line passes through the points $A(1, 2, 4)$ and $B(3, -1, -2)$. Find the equation of the line in a) vector format b) in parametric format and c) in symmetric format.

Solution:

a) For vector format we need to find a vector along the line. This can be[2]

[2] Note that it is also valid to take the vector $\overrightarrow{BA}$.

$$\begin{aligned} \overrightarrow{AB} &= (3-1)\,\hat{\imath} + (-1-2)\,\hat{\jmath} + (-2-4)\,\hat{k} \\ &= 2\hat{\imath} - 3\hat{\jmath} - 6\hat{k} \end{aligned}$$

The vector $\overrightarrow{AB}$ is the direction vector along the line. We can now use either point, A or B to find the equation of the line in vector format. So the possible vector forms are:

$$\vec{r} = \left(\hat{\imath} + 2\hat{\jmath} + 4\hat{k}\right) + t\left(2\hat{\imath} - 3\hat{\jmath} - 6\hat{k}\right) \tag{8}$$

where $t \in \mathbb{R}$ and we have used point A, or

$$\vec{r} = \left(3\hat{\imath} - \hat{\jmath} - 2\hat{k}\right) + t\left(2\hat{\imath} - 3\hat{\jmath} - 6\hat{k}\right) \tag{9}$$

where we have used point B. Either is acceptable.

b) For the parametric equations, you can use either (8) or (9). We will use (8). Equating components,[3] we have

$$\begin{aligned} \vec{r} &= \left(1\hat{\imath} + 2\hat{\jmath} + 4\hat{k}\right) + t\left(2\hat{\imath} - 3\hat{\jmath} - 6\hat{k}\right) \\ x\hat{\imath} + y\hat{\jmath} + z\hat{k} &= \left(1\hat{\imath} + 2\hat{\jmath} + 4\hat{k}\right) + t\left(2\hat{\imath} - 3\hat{\jmath} - 6\hat{k}\right) \end{aligned}$$

and so the parametric equations are

$$x = 1 + 2t \tag{10}$$
$$y = 2 - 3t \tag{11}$$
$$z = 4 - 6t \tag{12}$$

are the parametric equations.

c) The symmetric equations are obtained by solving $(10 - 12)$ for t to get

$$\begin{aligned} t &= \frac{x-1}{2} \\ t &= \frac{2-y}{3} \\ t &= \frac{4-z}{6}. \end{aligned}$$

So the symmetric equations are:

$$\frac{x-1}{2} = \frac{2-y}{3} = \frac{4-z}{6}.$$

Note that you will get different parametric and symmetric equations if you use (9) above. However they are acceptable solutions.

[3] Remember

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

where x, y and z are the coordinates of any point on the line.

Exercise:

Find the symmetric and parametric equations of the lines that satisfy the given conditions:

1. Contains the point $(1, -1, 2)$ with the direction vector $2\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$?
   Answer: $\frac{x-1}{2} = \frac{y+1}{-2} = \frac{z-2}{3}$ and $x = 1 + 2t$; $y = -1 - 2t$; $z = 2 + 3t$

2. Contains the point $(3, 4, -1)$ with the direction vector $\hat{\imath} + \hat{\jmath} + 5\hat{k}$?
   Answer: $\frac{x-3}{1} = \frac{y-4}{1} = \frac{z+1}{5}$ and $x = 3 + t$; $y = 4 + t$; $z = 5t - 1$

3. Contains the point $(2, 3, -1)$ parallel to $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{x+1}{3}$?
   Answer: $\frac{x-2}{2} = \frac{y-3}{-1} = \frac{z+1}{3}$ and $x = 2 + 2t$; $y = 3 - t$; $z = -1 + 3t$

4. Contains the point $(2, 2, 1)$ and $(1, 1, 3)$?
   Answer: $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-3}{-2}$ and $x = 1 + t$; $y = 1 + t$; $z = 3 - 2t$
   or
   $\frac{x-2}{1} = \frac{y-2}{1} = \frac{z-1}{-2}$ and $x = 2 + t; y = 2 + t; z = 1 - 2t$