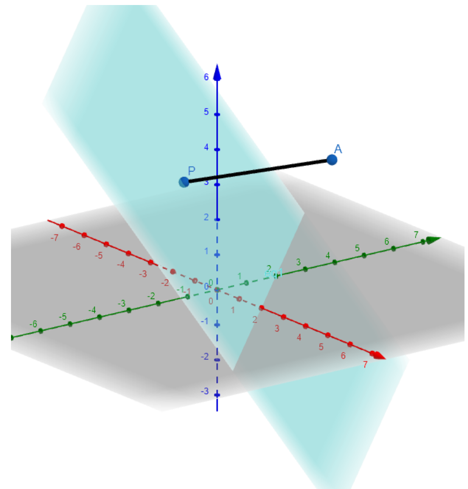
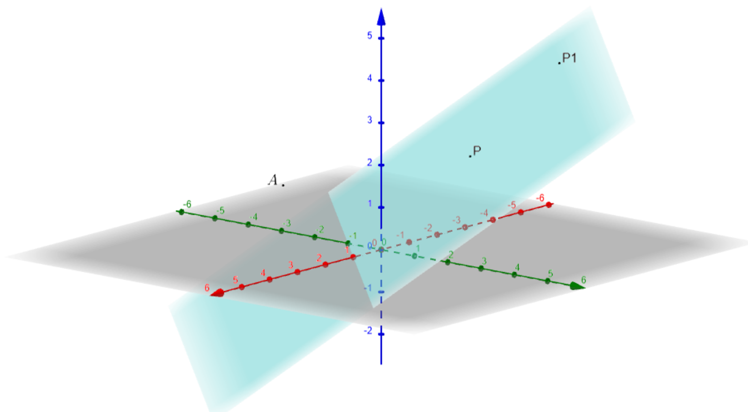
V10. Distance from a Point to a Plane



What do we mean when we talk about distance from a point to a plane? Consider the figure below. Here the $x-$axis is in red, the $y-$axis is in green and the $z-$axis is in blue. The grey plane is the $x-y$ plane, it extends in all directions. The aqua plane also extends in all directions.



You can have any number of distances from a point A to the aqua colored plane. For example, the distance from point A to point P is one possible answer while the distance from point A to point $P1$ is another. This makes no sense. The only interpretation of the question that provides a unique answer is "Find the shortest distance from a point to a plane". This means we want to find the perpendicular distance from the point to the plane.

When you are asked to find the distance from a point to a plane, you must find the shortest distance from a point to the plane.

Shortest Distance from a Point to a Plane

Consider the figure below. Here, the axes are exactly the same as the figure above. To find the shortest distance from a point A to the aqua colored plane we first choose any point on the plane P as shown below. Now define a vector from P to A, $\overrightarrow{PA}$. The shortest distance d is then[1]

$$d = \overrightarrow{PA} \cdot \hat{n} \qquad (1)$$

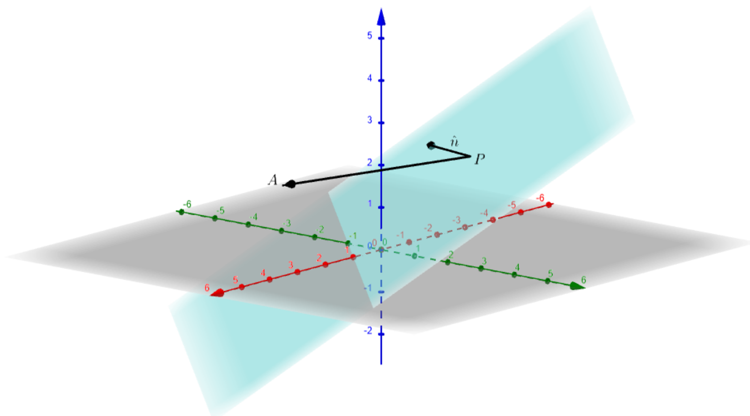
where $\hat{n}$ is a unit normal to the plane.

[1] This is sometimes called the scalar resolute of $\overrightarrow{PA}$ in the direction of $\hat{n}$. Since the unit vector

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

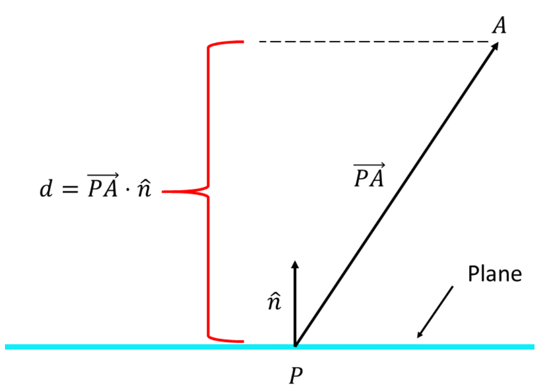
we can express the distance as

$$\begin{aligned} d &= \overrightarrow{PA} \cdot \hat{n} \\ &= \overrightarrow{PA} \cdot \frac{\vec{n}}{|\vec{n}|}. \end{aligned}$$



This may be easier to understand if we rotate the picture above so that we view the aqua plane "edge on" as shown below.[2]

[2] Note that if we take this viewpoint, the distance d does not change as we rotate the plane about the unit normal vector. This means the perpendicular distance d does not depend on how you view the problem.



The distance d is often referred to as the scalar resolute of $\overrightarrow{PA}$ in the direction of $\vec{n}$.

Unit Normal to a Plane

To get the distance from a plane to a point, we need to get a unit normal to the plane. For a plane

$$ax + by + cz = d$$

a normal vector is $a\hat{i} + b\hat{j} + c\hat{k}$. For example, the plane

$$2x + 3y - 6z = 20$$

has a normal

$$\vec{n} = 2\hat{i} + 3\hat{j} - 6\hat{k}.$$

To create a unit normal $\hat{n}$ we divide the normal $\vec{n}$ by it's length $|\vec{n}|$, [3]

$$\begin{aligned}|\vec{n}| &= \sqrt{n_1^2 + n_2^2 + n_3^2} \\ &= \sqrt{2^2 + 3^2 + (-6)^2} \\ &= \sqrt{49} \\ &= 7.\end{aligned}$$

[3] The length of a vector $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ is denoted $|\vec{n}|$ and

$$|\vec{n}| = \sqrt{n_1^2 + n_2^2 + n_3^2}.$$

So a unit normal to the plane is

$$\begin{aligned}\hat{n} &= \frac{\vec{n}}{|\vec{n}|} \\ &= \frac{1}{7}\left(2\hat{i} + 3\hat{j} - 6\hat{k}\right)\end{aligned}$$

Example

Find the distance from the point $(2, 5, 4)$ to the plane $x + 2y + 2z = 2$.

Solution:

Let point $A = (2, 5, 4)$. Now find a point on the plane[4]We take $P = (2, 0, 0)$. The vector from P to A is

$$\begin{aligned}\overrightarrow{PA} &= 2\hat{i} + 5\hat{j} + 4\hat{k} - \left(2\hat{i} + 0\hat{j} + 0\hat{k}\right) \\ &= 5\hat{j} + 4\hat{k}.\end{aligned}$$

[4] It can be ANY point. So to make it easy, we can take $y = z = 0$. Substituting in the equation for the plane we get $x = 2$. So the point $P = (2, 0, 0)$ is on the plane.

We now need to find a unit vector that is normal to the plane. A normal vector is

$$\vec{n} = \hat{i} + 2\hat{j} + 2\hat{k}.$$

A unit normal vector is

$$\begin{aligned}\hat{n} &= \frac{\vec{n}}{|\vec{n}|} \\ &= \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} \\ &= \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{9}} \\ &= \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right).\end{aligned}$$

Using (1) above, the distance d, from $A = (2, 5, 4)$ to the plane

$$x + 2y + 2z = 2$$

is

$$\begin{aligned} d &= \overrightarrow{PA} \cdot \hat{n} \\ &= \left(5\hat{j} + 4\hat{k}\right) \cdot \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \\ &= \frac{1}{3}\left(10 + 8\right) \\ &= 6. \end{aligned}$$

Exercises

Find the distance from the given point to the given plane:

1. $(0,0,0)$; $2x + 3y - z = 6$.
2. $(-1,1,2)$; $x + y = 2$.
3. $(1,2,3)$; $3x + 4y - z = 1$.

Answers

1. $\frac{6}{\sqrt{14}}$.
2. $\sqrt{2}$.
3. $\frac{7}{\sqrt{26}}$.