NM1: *Newton's Method*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

You are used to solving equations using basic algebraic operations and perhaps the quadratic formula. However, some equations cannot be solved using these methods. For example the equations:

$$e^{-x} = x \qquad (1)$$

or

$$-\ln(x) = x^3 \qquad (2)$$

cannot be solved using conventional methods. However they can be solved using numerical methods. The important aspect of equations (1) and (2) is that they can be re-arranged to $f(x) = 0$. That is, the problem can be transformed to finding the root[1] of a function $f(x)$.

[1] A root of a function $f(x)$ is the value of x for which $f(x) = 0$.

Common numerical methods for root finding are:

1. Newton's method (also called the Newton-Raphson method)

2. the Bisection method.

In this module we consider Newton's method.

The Newton-Raphson Method

Newton's method (Newton-Raphson) can provide the value of x such that $f(x) = 0$.

It has the form:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

where n starts at 0. The method is iterative, you start at $n = 0$ and continue by incrementing n by one until you achieve the desired accuracy.

Note that:

1. x_0 is called the first estimate of the root, or the initial guess,

2. x_{n+1} is the new estimate of the root, and x_n is the old estimate of the root,

3. $f(x_n)$ is the value of the function at x_n and,

4. $f'(x_n)$ is the value of the derivative of $f(x)$ at x_n.

Example 1

Find the solution of $e^{-x} = x$ for $0 < x < 1$ to three decimal places.

  Solution:

  First define the function:[2]

$$f(x) = e^{-x} - x.$$

[2] Note that you can also use $f(x) = x - e^{-x}$ as it is equal to zero at the same value of x as $f(x) = e^{-x} - x$.

We want to find x where $f(x) = 0$. To do this we need an initial guess x_0 of the root. This is easily done by graphing both $y = x$ and $y = e^{-x}$. The point where the graphs intersect is the point where $f(x) = 0$. The graphs are shown below.
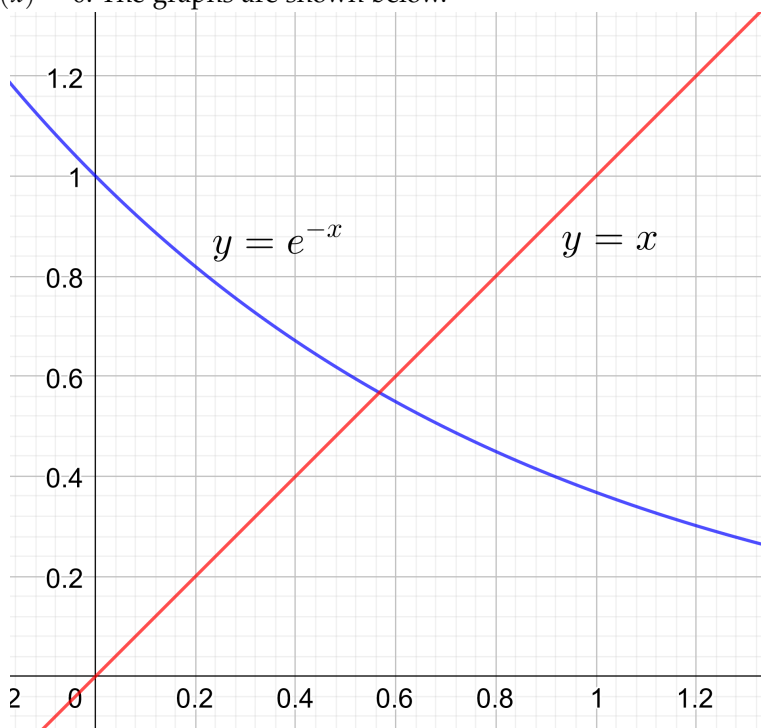
A suitable value for x_0 would be 0.5 and

$$f'(x) = -e^{-x} - 1$$

Using eqn (1), we can start the method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{e^{-0.5} - 0.5}{-e^{-0.5} - 1}$$

$$= 0.5663$$

$$x_2 = 0.5663 - \frac{f(0.5663)}{f'(0.5663)}$$

$$= 0.5663 - \frac{e^{-0.5663} - 0.5663}{-e^{-0.5663} - 1}$$

$$= 0.5671$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.5671 - \frac{e^{-0.5671} - 0.5671}{-e^{-0.5671} - 1}$$

$$= 0.5671.$$

There has been no change to the first four places so we stop and the answer is $x = 0.567$.

Example 2

Find the solution of $-\ln(x) = x^3$ correct to five decimal places with an initial guess of $x_0 = 1$.

Solution:

Define the function

$$f(x) = x^3 + \ln(x)$$

then

$$f'(x) = 3x^2 + \frac{1}{x}.$$

In this case we have a value for x_0 so there is no need to graph the functions. But we do it anyway.[3]

[3] It is a good idea to graph the functions because it allows us to get an idea of the answer. If Newton's method gives a result that is different to what we see in the graph, we have to consider that a mistake has been made or Newton's method has found another root that we did not see on our graph.

$y = -\ln(x)$ $y = x^3$
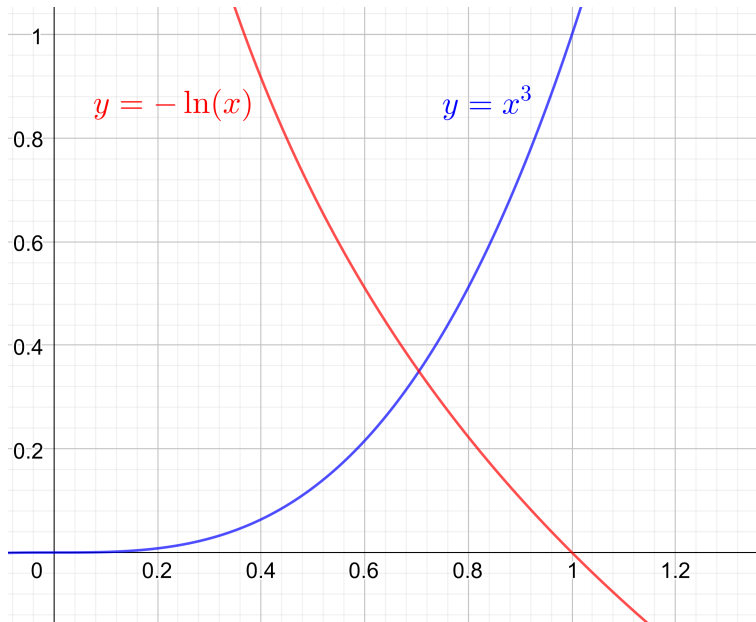
From the graph we see the answer should be about 0.7.

Using Newton's method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{1^3 + \ln(1)}{3 \cdot 1^2 + \left(\frac{1}{1}\right)}$$

$$= 1 - \frac{1 + 0}{3 + 1}$$

$$= 0.75$$

$$x_2 = 0.75 - \frac{(0.75)^3 + \ln(0.75)}{3 \cdot (0.75)^2 + \left(\frac{1}{0.75}\right)}$$

$$= 0.7055775$$

$$x_3 = 0.7055775 - \frac{(0.7055775)^3 + \ln(0.7055775)}{3 \cdot (0.7055775)^2 + \left(\frac{1}{0.7055775}\right)}$$

$$= 0.7047098$$

$$x_4 = 0.7047098 - \frac{(0.7047098)^3 + \ln(0.7047098)}{3 \cdot (0.7047098)^2 + \left(\frac{1}{0.7047098}\right)}$$

$$= 0.7047095.$$

As we have no change in the first five places after the decimal point we stop and take the answer as $x = 0.70471$. Note that we round up the fifth digit.

Convergence of the Newton-Raphson Method

Convergence indicates how quickly you will get an acceptable value of $f(x) = 0$. Slow convergence implies many iterations, while fast convergence means few iterations.

Convergence of the method depends on the function $f(x)$ and the initial guess. If the initial guess is in an area where the gradient of $f(x)$ is nearly zero, convergence can be slow.

Consider the function $f(x) = x^4 - 1$. It is graphed below.

$$f(x) = x^4 - 1$$
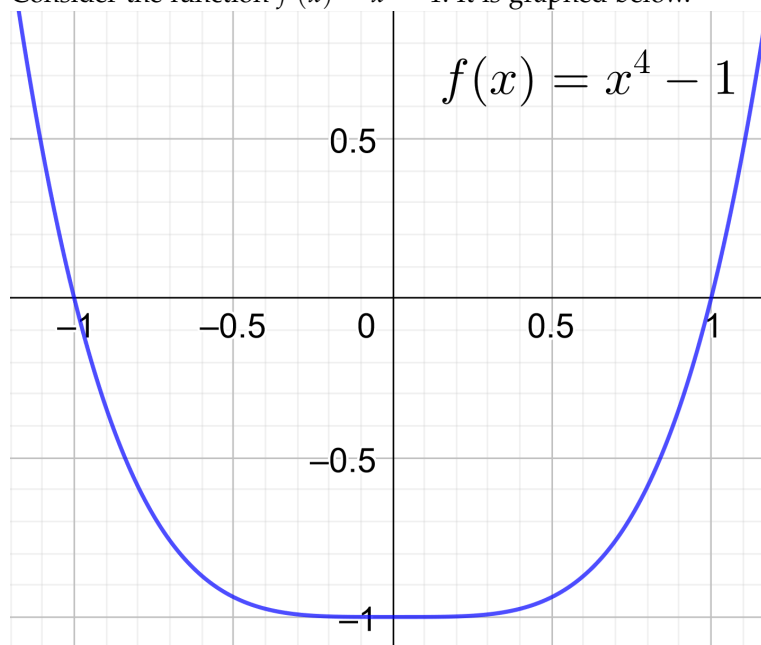
Obviously the solutions of $f(x) = 0$ are $x = -1$ and $x = 1$. These can be obtained by basic algebra and there is no need to use Newton's method. However we will use this function to see what happens with different guesses for x_0. We assume an accuracy to two decimal places and a calculation tolerance of 0.0000001.

Let $x_0 = 0.1$. In this region the gradient of $f(x)$ is close to zero. Newton's method takes 25 iterations to get the result $x = 1.00$.

Let $x_0 = 0.25$ then it takes 15 iterations to get the result $x = 1.00$.

Let $x_0 = 0.5$ then it takes 9 iterations to get the result $x = 1.00$.

Let $x_0 = 0.75$ then it takes 6 iterations to get the result $x = 1.00$.

So we see the number of iterations decrease (higher convergence) when the gradient of $f(x)$ increases.

Convergence to Another Root.

If the function $f(x)$ has several roots, Newton's method may converge to any of these. For example the function $f(x) = x^3 + 2x^2 -$

$5x - 6$ has three roots at $x = -3, -1$ and 2 as shown below.

$$f(x) = x^3 + 2x^2 - 5x - 6$$
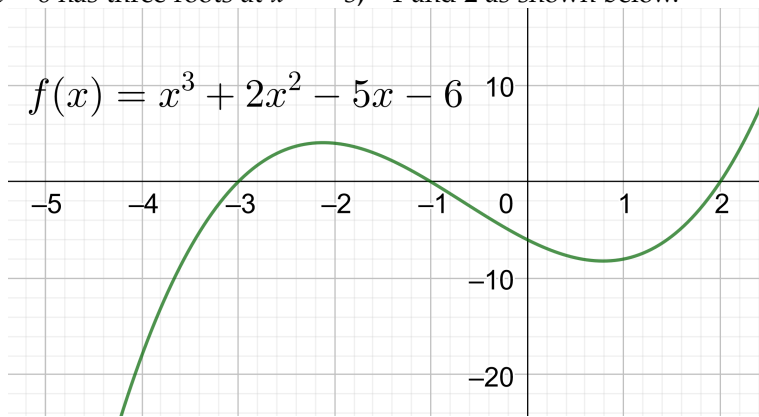
If we set $x_0 = -2$ we, perhaps, would think the method should find the root $x = -3$ or -1. But Newton's method finds the root at $x = 2$. This is why it is important to graph the function if possible.