

M10 Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are an important part of an engineer's mathematical toolbox. They give us an understanding of how buildings, structures, automobiles and materials react in real life. Moreover they are useful for data scientists.

This module does not go into each of these facets of eigenvalues and eigenvectors but describes the method for calculating them.

Definitions

Let A be an $n \times n$ matrix. ¹

Let X be an $n \times 1$ vector. ²

If for some constant λ we have

$$AX = \lambda X \quad (1)$$

then λ is called an eigenvalue of A and X is called an eigenvector of A . Note that X is a column vector of order $(n \times 1)$. We write

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Usually, if A is a 2×2 matrix, we write

$$X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If A is a 3×3 matrix, we write

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Finding Eigenvalues

Remember that the determinant of a matrix A may be written as $\det A$ or $|A|$.

$$|A - \lambda X| = 0$$

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

¹ The order of a matrix A is the number of rows in A times the number of columns in A . If A is a matrix with 5 rows and 6 columns, it is said to be a 5×6 (read as five times six) matrix, or has order 5×6 . Eigenvector and eigenvalue problems are restricted to square matrices that is, the number of rows and columns of the matrix must be equal.

² Note that X is a column matrix (also known as a column vector).

For a square matrix A the eigenvalues are found by solving the equation

$$\det(A - \lambda I) = |A - \lambda I| = 0. \quad (2)$$

1. Equation (2) is called the CHARACTERISTIC EQUATION of the matrix A . So to find eigenvalues, we solve the characteristic equation.
2. If A is an $n \times n$ matrix, there will be at most n distinct eigenvalues of A .

Example 1

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -4 & -2 \\ 3 & 3 \end{bmatrix}.$$

Solution:

The 2×2 identity matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$A - \lambda I = \begin{bmatrix} -4 & -2 \\ 3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} -4 & -2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad (4)$$

$$= \begin{bmatrix} -4 - \lambda & -2 \\ 3 & 3 - \lambda \end{bmatrix}. \quad (5)$$

Note that in practice we would not bother to include steps (3) and (4) above. We would just go directly to (5).

To find the eigenvalues we solve $\det[A - \lambda I] = 0$ ³. That is,

³ This is the same as

$$|A - \lambda I| = 0.$$

$$\begin{aligned} \det[A - \lambda I] &= 0 \\ \det \begin{bmatrix} -4 - \lambda & -2 \\ 3 & 3 - \lambda \end{bmatrix} &= 0 \\ (-4 - \lambda)(3 - \lambda) - 3(-2) &= 0 \\ -12 + 4\lambda - 3\lambda + \lambda^2 + 6 &= 0 \\ \lambda^2 + \lambda - 6 &= 0. \end{aligned}$$

The equation $\lambda^2 + \lambda - 6 = 0$ is the characteristic equation of the matrix A . Factorising the characteristic equation,⁴

$$\begin{aligned} 0 &= \lambda^2 - \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2) \end{aligned}$$

and so $\lambda = -3$ or $\lambda = 2$. The two eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 2$.

Example 2

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

Solution:

We have

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & -1 \\ 0 & 2 - \lambda \end{bmatrix}$$

and to find the eigenvalues we solve $\det[A - \lambda I] = 0$. That is

$$\begin{aligned} \det \begin{bmatrix} 2 - \lambda & -1 \\ 0 & 2 - \lambda \end{bmatrix} &= 0 \\ (2 - \lambda)(2 - \lambda) &= 0. \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 2$. Note there are two eigenvalues but they have the same value. These are called repeated eigenvalues.

Eigenvectors

To each eigenvalue of a matrix A there is a corresponding eigenvector.

To find the eigenvectors we solve equation (1) above for each eigenvalue. It is useful to rearrange equation (1) as follows

$$\begin{aligned} AX &= \lambda X \\ AX - \lambda X &= 0 \\ (A - \lambda I)X &= 0. \end{aligned} \tag{6}$$

Note that we don't write $(A - \lambda)X = 0$ because this makes no sense. A is an $n \times n$ matrix and λ is a scalar (a number). You can't subtract a number from a matrix and so we write $(A - \lambda I)X$ as in equation (6).

⁴ Sometimes it is not so easy to factorise the equation. In this case you can use the quadratic formula. The solution of the quadratic

$$ax^2 + bx + c = 0$$

is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To obtain eigenvectors, we solve equation (6)

$$(A - \lambda I) X = 0$$

for each eigenvalue λ .

Note that a multiple of any eigenvector is an eigenvector. We do not consider $\vec{X} = \vec{0}$ to be an eigenvector.

Example 3

Find the eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & -2 \\ 3 & 3 \end{bmatrix}.$$

From example 1 above we know the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 2$. We need to solve

$$(A - \lambda I) X = 0$$

to get the eigenvectors. Now

$$(A - \lambda I) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 - \lambda & -2 \\ 3 & 3 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad (7)$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

For $\lambda_1 = -3$ we have

$$\begin{bmatrix} -4 - (-3) & -2 \\ 3 & 3 - (-3) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

This gives two linear equations

$$-x - 2y = 0$$

$$3x + 6y = 0.$$

These cannot be solved to give a unique solution as one is a multiple of another⁵. Dividing the second equation by 3 gives

$$x + 2y = 0$$

$$x = -2y.$$

⁵ Multiplying the first equation by -3 gives the second. We have one equation and two unknowns. There will be an infinite number of solutions.

Let $y = t$, $t \in \mathbb{R}$ then the solution is $x = -2t$, $y = t$ and the eigenvector has the form

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -2t \\ t \end{bmatrix} \\ &= t \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

As t can be any number there are an infinite number of eigenvectors. However, the convention is that we recognise any eigenvector multiplied by a scalar is still an eigenvector and we state the eigenvector in the lowest possible terms. So, the eigenvector corresponding to $\lambda_1 = -3$ is

$$X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 2$, we have

$$\begin{aligned} \begin{bmatrix} -4-2 & -2 \\ 3 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= 0 \\ \begin{bmatrix} -6 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= 0. \end{aligned}$$

This gives two linear equations

$$\begin{aligned} -6x - 2y &= 0 \\ 3x + y &= 0. \end{aligned}$$

Again we see that the one equation is a multiple of another⁶. Let $x = t$ then from the second equation, $y = -3t$ and the eigenvector has the form

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} t \\ -3t \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \end{aligned}$$

In lowest terms, the eigenvalue corresponding to $\lambda_2 = 2$ is

$$X_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

See exercise 1 for some practice examples.

Eigenvalues and Eigenvectors for 3×3 Matrices

Regardless of the size of the matrix, the method is the same. For eigenvalues we solve equation (2) and for eigenvectors we solve equation (6).

⁶ Multiplying the first equation by -3 gives the second. We have one equation and two unknowns. There will be an infinite number of solutions.

Example 4

Find the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Solution:

First find the eigenvalues by solving:

$$\begin{aligned} 0 &= \det [B - \lambda I] \\ &= \det \begin{bmatrix} 1 - \lambda & 1 & -2 \\ -1 & 0 - \lambda & 1 \\ -2 & 1 & 1 - \lambda \end{bmatrix} \\ &= (1 - \lambda) [-\lambda(1 - \lambda) - 1(1)] - 1[-1(1 - \lambda) - 1(-2)] - 2[1(-1) - (-2)(-\lambda)] \\ &= (1 - \lambda)(-\lambda + \lambda^2 - 1) - (-1 + \lambda + 2) - 2(-1 - 2\lambda) \\ &= -\lambda + \lambda^2 - 1 + \lambda^2 - \lambda^3 + \lambda + 1 - \lambda - 2 + 2 + 4\lambda \\ &= -\lambda^3 + 2\lambda^2 + 3\lambda. \end{aligned}$$

The characteristic equation is:⁷

$$-\lambda^3 + 2\lambda^2 + 3\lambda = 0.$$

⁷Note that the characteristic equation is a cubic, that is it is a polynomial of degree 3. If we had a 4×4 matrix, the characteristic equation would be a polynomial of degree 4 and so on.

To solve this, we factorize to get:

$$\begin{aligned} -\lambda^3 + 2\lambda^2 + 3\lambda &= 0 \\ -\lambda(\lambda^2 - 2\lambda - 3) &= 0 \\ -\lambda(\lambda - 3)(\lambda + 1) &= 0. \end{aligned}$$

Hence the eigenvalues of B are: $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -1$. In many courses you may solve the characteristic equation using a CAS calculator. Check this with your teacher.

We now find the eigenvectors by solving equation (6) for each eigenvalue.

For $\lambda_1 = 0$, we have⁸

⁸Note that

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$(B - \lambda I)X = 0.$$

$$\begin{bmatrix} 1-0 & 1 & -2 \\ -1 & 0-0 & 1 \\ -2 & 1 & 1-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y - 2z = 0 \quad (8)$$

$$-x + z = 0 \quad (9)$$

$$-2x + y + z = 0. \quad (10)$$

Solution of these equations may be done manually or, in some courses, with a CAS calculator. Check with your teacher if you are allowed to solve them with a calculator. We solve them manually below.

From equation (9)

$$z = x.$$

Substituting this in (8) gives

$$x + y - 2x = 0$$

$$x = y.$$

Hence the solution is $x = y = z = t$ where $t \in \mathbb{R}$. Hence the eigenvector has the form

$$X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So the eigenvector corresponding to $\lambda_1 = 0$ is

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 3$ we have

$$(B - \lambda I) X = 0.$$

$$\begin{bmatrix} 1-3 & 1 & -2 \\ -1 & 0-3 & 1 \\ -2 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & -2 \\ -1 & -3 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y - 2z = 0 \quad (11)$$

$$-x - 3y + z = 0 \quad (12)$$

$$-2x + y - 2z = 0. \quad (13)$$

Equations (11) and (13) are identical and so we have to solve (11) and (12). Let $z = t$ where $t \in \mathbb{R}$ then (11) and (12) become

$$-2x + y = 2t \quad (14)$$

$$-x - 3y = -t. \quad (15)$$

Multiplying (15) by 2 and subtracting from (14) gives

$$7y = 4t$$

$$y = \frac{4}{7}t.$$

Multiplying (14) by 3 and adding to (15) gives

$$-7x = 5t$$

$$x = -\frac{5}{7}t.$$

Hence the eigenvector has the form

$$X = t \begin{bmatrix} -\frac{5}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix}.$$

So the eigenvector corresponding to $\lambda_2 = 3$ is (setting $t = 7$)

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}. \quad (16)$$

Note that we chose $t = 7$ to get rid of the fraction in the eigenvector.
 Note also that the eigenvector

$$X = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$$

is also correct as it is just (16) multiplied by -1 .

For $\lambda_3 = -1$ we have

$$(B - \lambda I)X = 0.$$

$$\begin{bmatrix} 1 - (-1) & 1 & -2 \\ -1 & 0 - (-1) & 1 \\ -2 & 1 & 1 - (-1) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$2x + y - 2z = 0 \quad (17)$$

$$-x + y + z = 0 \quad (18)$$

$$-2x + y + 2z = 0. \quad (19)$$

Adding (17) and (19) we get

$$2y = 0$$

$$y = 0.$$

Substituting this into (18) gives

$$-x + y + z = 0$$

$$x = z.$$

Let $z = t, t \in \mathbb{R}$ then the eigenvector has the form

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the eigenvector to $\lambda_3 = -1$ is

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To summarise, the eigenvalues of B are $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -1$.
The corresponding eigenvectors are

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

respectively.

Example 5

Are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors of $A = \begin{bmatrix} 1 & -5 & 8 \\ 1 & -2 & 1 \\ 2 & -1 & -5 \end{bmatrix}$?

Solution:

One way to solve this problem is to do as in Example 4 above and calculate all the eigenvalues and their corresponding eigenvectors. However, this is the long way. A shorter and better way is to use the definition of the eigenvector in equation (1) above.⁹

For X_1 we have from the left hand side of equation (1):

$$\begin{aligned} &^9 \text{ Recall equation (1) is} \\ &AX = \lambda X. \end{aligned}$$

$$\begin{aligned} AX_1 &= \begin{bmatrix} 1 & -5 & 8 \\ 1 & -2 & 1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 1 - 5 \times 2 + 8 \times 1 \\ 1 \times 1 - 2 \times 2 + 1 \times 1 \\ 2 \times 1 - 1 \times 2 - 5 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix} \end{aligned} \tag{20}$$

Now consider the right hand side of equation (1). If X_1 is an eigenvector, there exists a number λ such that, from (20),

$$\begin{aligned} \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix} &= \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ 2\lambda \\ \lambda \end{bmatrix}. \end{aligned} \tag{21}$$

The first two rows of (21) imply that $\lambda = -1$. But then the third row implies that $-5 = -1$ which is a contradiction. We conclude that X_1 is not an eigenvector of A .

For X_2 we have from the left hand side of equation (1):

$$\begin{aligned} AX_1 &= \begin{bmatrix} 1 & -5 & 8 \\ 1 & -2 & 1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times (-1) - 5 \times 0 + 8 \times 1 \\ 1 \times (-1) - 2 \times 0 + 1 \times 1 \\ 2 \times (-1) - 1 \times 0 - 5 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}. \end{aligned} \tag{22}$$

Now consider the right hand side of equation (1). If X_2 is an eigenvector, there exists a number λ such that, from (22),

$$\begin{aligned} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} &= \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\lambda \\ 0 \\ \lambda \end{bmatrix}. \end{aligned}$$

It is clear that $\lambda = -7$ and so $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix A .

Important Points to Remember

- An $n \times n$ matrix will have at most n distinct eigenvalues.
- An eigenvalue may be zero.
- An eigenvector multiplied by any number other than 0 is also an eigenvector. In particular, if X is an eigenvector then so is $-X$.
- An eigenvector cannot be the zero vector, $\vec{0}$.

See exercises 2 and 3 for some practice problems.

Exercise 1

Find the eigenvalues and eigenvectors for the following matrices:

$$\begin{aligned} \text{a)} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \text{b)} & \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \\ \text{c)} & \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

Answers

For exercise 1.

$$\text{a) } \lambda_1 = 3, \lambda_2 = 1$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\text{b) } \lambda_1 = 3, \lambda_2 = 3 \text{ (repeated root)}$$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\text{c) } \lambda_1 = 2, \lambda = 0$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Exercise 2

Which of the following are eigenvectors of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$?

$$\text{a) } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{c) } \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Answers

For exercise 2.

Answer is a). Remember b) is a zero vector and so can't be an eigenvector even though it satisfies equation (1).

Exercise 3

Find the eigenvalues and eigenvectors for the following matrices:

$$a) \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & -1 \end{bmatrix} \quad b) \begin{bmatrix} -2 & 0 & -1 \\ 0 & 5 & 2 \\ 3 & 0 & 2 \end{bmatrix}$$

Answers

For exercise 3.

a.

$$\lambda_1 = 2, X_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1, X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\lambda_3 = 1, X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

b.

$$\lambda_1 = 5, X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1, X_2 = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}$$

$$\lambda_3 = 1, X_3 = \begin{bmatrix} -2 \\ -3 \\ 6 \end{bmatrix}.$$