# LT3 LAPLACE TRANSFORMS – DIFFERENTIAL EQUATIONS

## Solving Differential Equations Using Laplace Transforms

### *Example*

Given the following first order differential equation, $\frac{dy}{dx} + y = 3e^{2t}$, where $y(0)=4$.

Find $y(t)$ using Laplace Transforms.

Sol$^{\underline{n}}$:

To begin solving the differential equation we would start by taking the Laplace transform of both sides of the equation.

| | |
|---|---|
| $L\left[\frac{dy}{dt} + y\right] = L\left[3e^{2t}\right]$ | Taking the Laplace Transform of both sides of the equation. |
| $L\left[\frac{dy}{dt}\right] + L[y] = 3L\left[e^{2t}\right]$ | Separating terms. |
| $sY - y(0) + Y = 3 \times \frac{1}{s-2}$ | Transforms as derived from tables. |
| $sY - 4 + Y = \frac{3}{s-2}$ | Substituting for $y(0)=4$ |
| $Y(s+1) = \frac{3}{s-2} + 4$ | Taking $Y$ as a common factor. |
| $Y = \frac{3}{(s-2)(s+1)} + \frac{4}{(s+1)}$ | Making $Y$ the subject. |

$$Y = \frac{3}{(s+1)(s-2)} + \frac{4(s-2)}{(s+1)(s-2)}$$

$$Y = \frac{4s-5}{(s-2)(s+1)}$$

Use partial fractions to expand $\frac{4s-5}{(s-2)(s+1)}$

$$\therefore \quad \frac{4s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$4s - 5 = A(s+1) + B(s-2)$$

By selecting appropriate values of $s$, we can solve for $A$ & $B$.

Letting $s = -1$, and substituting into the above equation gives

$$4(-1)-5 = A(-1+1)+B(-1-2)$$
$$-4-5 = A(0)+B(-3)$$
$$-9 = -3B$$
$$B = \frac{-9}{-3} = 3$$

Now let $s = 2$, and substitute into the same equation

$$4(2)-5 = A(2+1)+B(2-2)$$
$$8-5 = A(3)+B(0)$$
$$3 = 3A$$
$$A = \frac{3}{3} = 1$$

So

$$\frac{4s-5}{(s-2)(s+1)} = \frac{1}{s-2}+\frac{3}{s+1}$$

Therefore

$$Y = \frac{1}{s-2}+\frac{3}{s+1}$$

To obtain a solution $y(t)$ to the differential equation from $Y(s)$ we need to find the inverse Laplace transform of $Y$.

$$\therefore L^{-1}[Y] = L^{-1}\left[\frac{1}{s-2}+\frac{3}{s+1}\right]$$

$$\therefore y(t) = L^{-1}\left[\frac{1}{s-2}\right]+3L^{-1}\left[\frac{1}{s+1}\right]$$

Inverse transforms obtained from tables.

$$\therefore y(t) = e^{2t}+3e^{-t}$$

## Example

Given the following second order differential equation, $y'' + y' = 5\cos 2t$ ; $y(0) = 0$; $y'(0) = 0$

Find $y(t)$ using Laplace Transforms.

Sol$^{\underline{n}}$:

$$L[y''] + L[y'] = L[5\cos 2t]$$

$$s^2Y - sy(0) - y'(0) + sY - y(0) = \frac{5s}{s^2 + 2^2}$$

$$Y(s^2 + s) = \frac{5s}{s^2 + 4}$$

$$Y = \frac{5s}{(s^2 + s)(s^2 + 4)}$$

$$Y = \frac{5s}{(s)(s+1)(s^2 + 4)} = \frac{5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4}$$

$$\therefore \qquad 5 = A(s^2 + 4) + (Bs + C)(s + 1)$$

An alternative method for solving the unknowns *A, B, & C* in the above equation is called "Equating coefficients of powers of s":

| | LHS | = | RHS | |
|---|---|---|---|---|
| $s^0$ : | 5 | = | $4A + C$ | eq$^{\underline{n}}$ 1. |
| $s^1$ : | 0 | = | $B + C$ | eq$^{\underline{n}}$ 2. |
| $s^2$ : | 0 | = | $A + B$ | eq$^{\underline{n}}$ 3. |

From eq$^{\underline{n}}$ 3 $\qquad A = -B$

From eq$^{\underline{n}}$ 2 $\qquad C = -B$

Substitute in eq$^{\underline{n}}$ 1 $\qquad 5 = 4(-B) + (-B)$

$$\therefore \qquad A = 1, \quad B = -1, \quad C = 1$$

$$\therefore \qquad Y = \frac{1}{s+1} + \frac{-s+1}{s^2+4} = \frac{1}{s+1} - \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$\therefore \qquad y(t) = L^{-1}[Y] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right]$$

From tables:

$$y(t) = e^{-t} - \cos 2t + \sin 2t$$

## Exercise

Using partial fractions determine the inverse Laplace Transforms of the following expressions.

a. $\frac{5s+2}{(s+1)(s+2)}$ b. $\frac{3s+4}{(s+2)(s+3)}$ c. $\frac{4s+1}{(s+3)(s+4)}$

d. $\frac{6s-5}{(s+5)(s+3)}$ e. $\frac{4s+1}{s(s+2)(s+3)}$ f. $\frac{2s-8}{(s+2)(s^2+7s+6)}$

## Answers

a. $8e^{-2t} - 3e^{-t}$ b. $5e^{-3t} - 2e^{-2t}$ c. $15e^{-4t} - 11e^{-3t}$

d. $\frac{35}{2}e^{-5t} - \frac{23}{2}e^{-3t}$ e. $\frac{1}{6} + \frac{7}{2}e^{-2t} - \frac{11}{3}e^{-3t}$ f. $3e^{-2t} - e^{-6t} - 2e^{-t}$

## Exercise

Solve the differential equations using Laplace Transform methods.

a. $\frac{dx}{dt} + x = 0$ ; $x(0) = 3$

b. $\frac{dx}{dt} + x = 9e^{2t}$ ; $x(0) = 3$

c. $y'' - y = -t^2$
$y(0) = 2$ ; $y'(0) = 0$

d. $y'' - y' - 2y = -2$
$y(0) = 2$ ; $y'(0) = 0$

e. $x'' + 2x' + 2x = e^{-t}$
$x(0) = x'(0) = 0$

f. $x'' - 5x' + 6x = 6t - 4$
$x(0) = 1$ ; $x'(0) = 2$

## Answers

a. $x(t) = 3e^{-t}$

b. $x(t) = 3e^{2t}$

c. $y(t) = 2 + t^2$

d. $y(t) = 1 + \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$

e. $x(t) = e^{-t} - e^{-t}\cos t$

f. $x(t) = \frac{1}{6} + t + \frac{3}{2}e^{2t} - \frac{2}{3}e^{3t}$