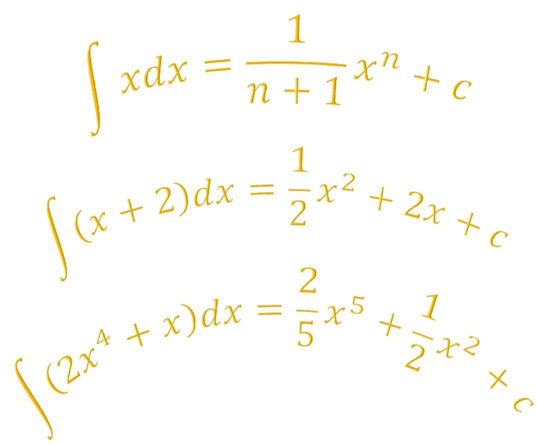
IN2: Integration of Polynomials



This module shows how to integrate functions like :

$$f(x) = 2x^4 - 3x + 2$$

or more generally:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where a_i, $i = 1, 2, 3, \ldots, n$ are constants. Such functions are called polynomials.

Antidifferentiation

Antidifferentiation is the opposite process to differentiation. For example if $y = x^2$, then an antiderivative of y is $\frac{1}{3}x^3$ because the derivative of $\frac{1}{3}x^3$ is y. A mathematical notation for this is

$$\int x^2 dx = \frac{1}{3}x^3 + c, \text{ where } c \text{ is a constant.}$$

This is read as "the indefinite integral of x^2 with respect to x is equal to $\frac{1}{3}x^3 + c$". The $\int$ sign means indefinite integral and dx means with respect to x. Here the words "indefinite integral" are the same as "antiderivative".

Integration is more complicated than differentiation. However there are some rules to help us. One of the most important is the power rule which says:[1]

Provided $n \neq 1$ then $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, where c is a constant.

[1] Note that if $n = 1$, $1/(n-1)$ would be $1/0$ which has no meaning. n can be any number (other than 1) - positive, negative or a fraction.

Instead of saying c is a constant, we sometimes write $c \in \mathbb{R}$ which means c is a real number.

Examples

$$\int xdx = \frac{1}{2}x^2 + c, \quad c \in \mathbb{R}$$
$$\int x^2 dx = \frac{1}{3}x^3 + c, \quad c \in \mathbb{R}$$

Linearity

The integral has two properties that let us evaluate more complex functions. They are called the linearity rules:

A constant that multiplies a function may be taken out of the integral. For a constant a and functions $f(x)$ and $g(x)$

$$\int af(x)\,dx = a\int f(x)\,dx$$

and

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$
$$\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx.$$

Examples

1. Integrate the constant 3.
   Solution:[2]

$$\begin{aligned}\int 3dx &= \int 3x^0dx \\ &= 3\int x^0dx \quad \text{by linearity} \\ &= 3x+c,\ c \text{ a constant, using the power rule.}\end{aligned}$$

[2] Note that $3 = 3x^0$. In fact, any constant m can be written as mx^0

2. Find $\int 5xdx$.
   Solution:

$$\begin{aligned}\int 5xdx &= 5\int xdx \quad \text{by linearity} \\ &= \frac{5}{2}x^2 + c\, c \text{ a constant, using the power rule.}\end{aligned}$$

3. Find $\int \left(5x^2-3x+4\right)dx$.
   Solution:[3]

$$\begin{aligned}\int \left(5x^2-3x+4\right)dx &= \int 5x^2dx + \int (-3x)\,dx \\ &\quad + \int 4dx \quad \text{using linearity} \\ &= 5\int x^2dx - 3\int xdx + 4\int dx \quad \text{using linearity} \\ &= \frac{5}{3}x^3 - \frac{3}{2}x^2 + 4x + c,\ c \text{ a constant, using the power rule.}\end{aligned}$$

[3] Note there is a constant for each of the integrals on the right hand side. However, a constant plus a constant still gives you a constant so we just use a single constant $c \in \mathbb{R}$.

Note that you can have any number of terms in the integral. For example if $a_1, a_2, a_3 \ldots a_n$ are constant

$$\int \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n\right) dx = \int a_0 dx + \int a_1 x dx + \int a_2 x^2 + \ldots + \int a_n x^n dx$$
$$= a_0 x + a_1 \int x dx + a_2 \int x^2 + \ldots + a_n \int x^n dx.$$

Examples

1. Integrate $f(x) = 13x^2 + x + 5$ with respect to x.
   Solution:
   $$\int \left(13x^2 + x + 5\right) dx = \frac{13}{3}x^3 + \frac{1}{2}x^2 + 5x + c \quad c \in \mathbb{R}.$$

2. Integrate $f(y) = 4y^5 - y^3$ with respect to y.
   Solution:
   $$\int \left(4y^5 - y^3\right) dy = \int 4y^5 dy - \int y^3 dy$$
   $$= \frac{4}{6}y^6 + \frac{1}{4}y^4 + c \quad \text{where } c \text{ is a constant}$$
   $$= \frac{2}{3}y^6 + \frac{1}{4}y^4 + c$$

3. Integrate $f(x) = 2x^7 + 2x^2 - x - 3$ with respect to x.
   Solution:
   $$\int \left(2x^7 + 2x^2 - x - 3\right) dx = \int 2x^7 dx + \int 2x^2 dx - \int x dx - \int 3 dx$$
   $$= \frac{2}{8}x^8 + \frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x + c \quad \text{where } c \text{ is a constant}$$
   $$= \frac{1}{4}x^8 + \frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x + c.$$

Exercises

Integrate the following with respect to x:

a)	x^3	*b*)	$x^3 - 2x$	*c*)	$2x^4 + 5x^2$	*d*)	$3x^2 - 7x^5$
e)	$x^2 - 2 + x$	*f*)	$2x + 2$	*g*)	$-6x^3 + 5x + 2$	*h*)	$9x^6 - 3x - 4$

Answers

a) $\frac{1}{4}x^4 + c$ *b*) $\frac{1}{4}x^4 - x^2 + c$ *c*) $\frac{2}{5}x^5 + \frac{5}{3}x^3 + c$ *d*) $x^3 - \frac{7}{6}x^6 + c$

e) $\frac{1}{3}x^3 - 2x + \frac{1}{2}x^2 + c$ *f*) $x^2 + 2x + c$ *g*) $-\frac{3}{2}x^4 + \frac{5}{2}x^2 + 2x + c$ *h*) $\frac{9}{7}x^7 - \frac{3}{2}x^2 - 4x + c$

The Definite Integral

Answers to all the examples above were functions of x and involved a constant. Such integrals are called indefinite integrals or antiderivatives. They do not have a specific value.

For a function $f(x)$ suppose we can write an antiderivative $F(x) = \int f(x)dx$. We then define a definite integral of a function $f(x)$ from a to b with respect to x using the notation:[4]

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where we assume $f(x)$ is defined for all x in the interval $[a, b]$. We call a the lower limit of integration and b the upper limit of integration. The notation $[F(x)]_a^b$ means substitute $x = b$ and $x = a$ into $F(x)$ and then subtract the values. This gets rid of the constant of integration c.

[4] This is known as the first fundamental theorem of calculus.

Examples of Definite Integrals

1. Suppose we want to find, $\int_1^3 xdx$. Then [5]

$$\begin{aligned}\int_1^3 xdx &= \left[\frac{1}{2}x^2 + c\right]_{x=1}^{x=3}, \quad c \in \mathbb{R} \\ &= \left(\frac{1}{2}3^2 + c\right) - \left(\frac{1}{2}1^2 + c\right) \\ &= \left(\frac{9}{2} + c\right) - \left(\frac{1}{2} + c\right) \\ &= \frac{8}{2} \\ &= 4.\end{aligned}$$

Note that the indefinite integral has a numerical value, in this case 4, and there is no constant c.

[5] First we find an antiderivative or indefinite integral $\int xdx + c$ and then evaluate this at the limits of integration. An antiderivative is $\frac{1}{2}x^2 + c$.

2. Calculate $\int_0^4 \left(x^3 - 6x^2 + x - 2\right) dx$.
   Solution:

$$\begin{aligned}\int_0^4 \left(x^3 - 6x^2 + x - 2\right) dx &= \left[\frac{1}{4}x^4 - 2x^3 + \frac{1}{2}x^2 - 2x\right]_{x=0}^{x=4} \\ &= \frac{1}{4}\left(4^4\right) - 2\left(4^3\right) + \frac{1}{2}\left(4^2\right) - 2(4) \\ &= 4^3 - 2(64) + \frac{1}{2}(16) - 8 \\ &= -64\end{aligned}$$

3. Evaluate $\int_{-1}^2 \left(5x^3 - 2x^2\right) dx$.

Solution:

$$\int_{-1}^{2} \left(5x^3 - 2x^2\right) dx = \left[\frac{5}{4}x^4 - \frac{2}{3}x^3\right]_{x=-1}^{x=2}$$
$$= \frac{5}{4}\left(2^4\right) - \frac{2}{3}\left(2^3\right) - \left(\frac{5}{4}(-1)^4 - \frac{2}{3}(-1)^3\right)$$
$$= \frac{5}{4}(16) - \frac{2}{3}(8) - \left(\frac{5}{4} + \frac{2}{3}\right)$$
$$= 20 - \frac{16}{3} - \frac{23}{12}$$
$$= \frac{153}{12}$$
$$= \frac{51}{4}$$

Exercises on Definite Integrals

Evaluate the following indefinite integrals:

a) $\int_1^2 (6x - 2)\, dx$ b) $\int_2^3 \left(8x^3 - 9x^2 - 2x\right) dx$ c) $\int_0^2 \left(7x^2 + 2\right) dx$

d) $\int_{-1}^2 \left(7x^3 - 6x + 3\right) dx$ e) $\int_{-1}^1 \left(8x^3 - x^2 + 6\right) dx$ f) $\int_{-1}^0 \left(8x^3 - x^2 + 6\right) dx$

Answers

a) 7 b) 68 c) 69 d) 105/4 e) 38/3 f) 11/3