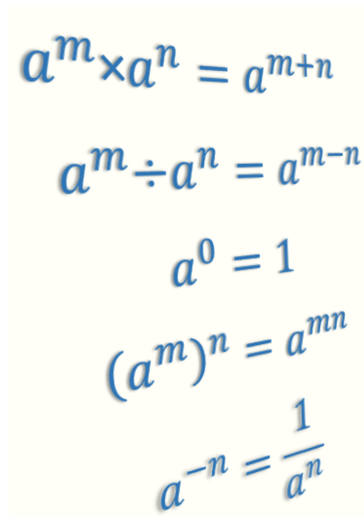
ILS 1.1 Indices



This module introduces rules for multiplying and dividing expressions using index notation. For example how to simplify expressions like $4a^3b \times 3ab^5$ or $9a^3b^2c \div 3ab^5$. We do not consider fractional indices which are covered in a different module. The plural of index is indices.

Click here to view a video showing how the rules work.

Index Notation

Consider the following examples:

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$
$$5^3 = 5 \times 5 \times 5 = 125$$
$$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$$

In general:

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}}$$

The letter n in a^n is referred to in one of three ways:

1. n is the index in a^n with a known as the base.
2. n is the exponent or power to which the base a is raised.
3. n is the logarithm, with a as the base. (see the Logarithms module)

When a number such as 125 is written in the form 5^3 we say it is written as an exponential or in index notation. Multiplication and division of numbers or expressions written in index notation is achieved using **index laws**.

Index Laws

This section states and gives examples of universal index laws.

First Index Law

To multiply index expressions you add the indices. For example:

$$\begin{aligned} 2^3 \times 2^2 &= (2 \times 2 \times 2) \times (2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^5 \end{aligned}$$

Therefore $2^3 \times 2^2 = 2^{3+2} = 2^5$. In general:

First Index Law:
$$a^m \times a^n = a^{m+n}$$

Second Index Law

To divide expressions subtract the indices. For example:

$$\begin{aligned} 3^5 \div 3^3 &= \frac{3^5}{3^3} \\ &= \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3} \\ &= \frac{3 \times 3}{1} \quad \text{cancelling three lots of 3} \\ &= 3^2 \end{aligned}$$

Therefore $3^5 \div 3^3 = \frac{3^5}{3^3} = 3^{5-3} = 3^2$. In general:

Second Index Law:
$$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Note that expressions in index form can only be multiplied or divided if they have the same base.

Third Index Law

To raise an expression in index form to a power, multiply the indices. For example:

$$\begin{aligned} \left(5^2\right)^3 &= 5^2 \times 5^2 \times 5^2 \\ &= 5^{2+2+2} \quad \text{using the first index law} \\ &= 5^6 \end{aligned}$$

Therefore $\left(5^2\right)^3 = 5^{2\times 3} = 5^6$. In general:

Third Index Law:
$$(a^m)^n = a^{m\times n}$$

This also leads to the expression:

$$(a^m b^p)^n = a^{mn} b^{pn}$$

Be careful as this is true for multiplication and division only , not addition or subtraction , so that $(a+b)^n \neq a^n + b^n$

Examples of Index Laws

1. Simplify $x^5 \times x^6$.
   Solution:

$$\begin{aligned} x^5 \times x^6 &= x^{5+6} \quad \text{by the first law} \\ &= x^{11} \end{aligned}$$

2. Simplify $a^5 \div a^3$.
   Solution:

$$\begin{aligned} a^5 \div a^3 &= \frac{a^5}{a^3} = a^{5-3} \quad \text{by the second law} \\ &= a^2 \end{aligned}$$

3. Simplify $\left(c^3\right)^4$.
   Solution:

$$\begin{aligned} \left(c^3\right)^4 &= c^{3\times 4} \quad \text{by the third law} \\ &= c^{12} \end{aligned}$$

4. Simplify $\left(2x^2\right)^3$.
   Solution:

$$\begin{aligned} \left(2x^2\right)^3 &= 2^3 \left(x^2\right)^3 \\ &= 8x^{2\times 3} \quad \text{by the third law} \\ &= 8x^6 \end{aligned}$$

Note that terms with different bases must be considered seperately when using the index laws , such as $(2a^3b^2)^4 = 2^4 a^{12} b^8$

Exercise 1 provides practice for these laws.

Zero Index

So far we have only considered expressions in which each index is a positive whole number[1]. The index laws also apply if the index is zero, negative or a fraction (fractional indices will be dealt with in another module).

[1] Whole numbers are called integers and positive whole numbers are called the positive integers.

Consider $2^3 \div 2^3 = \frac{2^3}{2^3} = \frac{2\times2\times2}{2\times2\times2} = \frac{8}{8} = 8 \div 8 = 1$. Using the second law, $2^3 \div 2^3 = 2^{3-3} = 2^0$

therefore $1 = 2^0$. In general any expression with a zero index is equal to 1. Also note that 0^0 is ambiguous and so we don't allow $a = 0$ in this law.

Zero law of indices:

$$a^0 = 1, \qquad a \neq 0$$

Examples of the Zero Index Law

$7^0 = 1 \qquad (xy)^0 = 1 \qquad \left(\frac{1}{2}\right)^0 = 1 \qquad \left(28x^2\right)^0 = 1$

Negative Indices

Consider $2^0 \div 2^4$.

$$\begin{aligned} 2^0 \div 2^4 &= \frac{2^0}{2^4} \quad \text{remember that } 2^0 = 1 \\ &= \frac{1}{2^4}. \end{aligned}$$

But

$$\begin{aligned} 2^0 \div 2^4 &= 2^{0-4} \quad \text{using the second law} \\ &= 2^{-4}. \end{aligned}$$

So $2^0 \div 2^4 = 2^{-4}$ and $2^0 \div 2^4 = \frac{1}{2^4}$

therefore $2^{-4} = \frac{1}{2^4}$. In general[2],

[2] Note that $1/0$ is undefined and so $a \neq 0$ in the law below.

$$a^{-n} = \frac{1}{a^n} \text{ which also leads to } \frac{1}{a^{-n}} = a^n, \qquad a \neq 0$$

Examples of Negative Indices

1. $2^{-3} = \frac{1}{8}$
2. $\frac{1}{x} = x^{-1}$
3. $2y^{-1} = \frac{2}{y}$
4. $\frac{1}{3x^{-2}} = \frac{x^2}{3}$
5. $\frac{1}{(-2a)^{-3}} = (-2a)^3$
6. $5ab^{-4} = \frac{5a}{b^4}$

Exercise 2 provides practice on zero and negative indices.

Summary of Index Laws

The following laws should be remembered.

Summary of Index Laws

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$, $\quad a \neq 0$
3. $(a^m)^n = a^{mn}$
4. $a^0 = 1$, $\quad a \neq 0$
5. $a^{-n} = \frac{1}{a^n}$, $\quad a \neq 0$

Combing of Index Laws

Index laws may be used to simplify complex expressions.

Examples

1. Simplify $\left(4a^2b\right)^3 \div b^2$.
   Solution:

$$\begin{aligned}\left(4a^2b\right)^3 \div b^2 &= \left(4^3a^6b^3\right) \div b^2 \quad \text{using law 3} \\ &= 4^3a^6b^1 \quad \text{using law 2} \\ &= 4^3a^6b \\ &= 64a^6b\end{aligned}$$

2. Simplify $\left(\frac{3a^3b}{c^2}\right)^2 \div \left(\frac{ab}{3c^{-2}}\right)^{-3}$ remember that $a \div \frac{b}{c} = a \times \frac{c}{b}$
   Solution:

$$\begin{aligned}\left(\frac{3a^3b}{c^2}\right)^2 \div \left(\frac{ab}{3c^{-2}}\right)^{-3} &= \frac{3^2a^6b^2}{c^4} \div \frac{a^{-3}b^{-3}}{3^{-3}c^6} \quad \text{by law 3} \\ &= \frac{3^2a^6b^2}{c^4} \times \frac{3^{-3}c^6}{a^{-3}b^{-3}} \quad \text{inverting the last term and multiplying} \\ &= \frac{3^{2-3}a^6b^2c^6}{c^4a^{-3}b^{-3}} \quad \text{by law 1} \\ &= 3^{-1}a^{6-(-3)}b^{2-(-3)}c^{6-4} \quad \text{by law 2} \\ &= 3^{-1}a^9b^5c^2 \quad \text{simplifying} \\ &= \frac{a^9b^5c^2}{3} \quad \text{by negative index law}\end{aligned}$$

3. Write $x^{-1} + x^2$ as a single fraction.

Solution:

$$\begin{aligned} x^{-1}+x^2 &= \frac{1}{x}+x^2 \quad \text{by negative index law} \\ &= \frac{1}{x}+\frac{xx^2}{x} \\ &= \frac{1+xx^2}{x} \quad \text{using a common denominator} \\ &= \frac{1+x^3}{x} \quad \text{using law 1} \end{aligned}$$

Exercise 1

Simplify the following:

a). $c^5 \times c^3 \times c^7$ b). $3 \times 2^2 \times 2^3$ c). $a^3 \times a^2b^3 \times ab^4$

d). $3^6 \div 3^4$ e). $a^8 \div a^3$ f). $x^4y^6 \div x^2y^3$

g). $\left(x^3\right)^4$ h). $\left(x^my^n\right)^5$

Answers to Exercise 1

a) c^{15} *b)* $3 \times 2^5 = 96$ *c)* a^6b^7 *d)* $3^2 = 9$
e) a^5 *f)* x^2y^3 *g)* x^{12} *h)* $x^{5m}y^{5n}$

Exercise 2

Write with positive indices and evaluate if possible:

a). x^{-6} b). 250^0 c). $3ab^{-5}$ d). $(pq)^{-2}$

e). $(5xy)^{-3}$ f). $\frac{2y}{z^{-5}}$ g). 2^{-5} h). $(-2)^{-3}$

i). $-(3^{-2})$ j). $2 \times (-5)^{-2}$

Answers to Exercise 2

a) $\frac{1}{x^6}$ *b)* 1 *c)* $\frac{3a}{b^5}$. *d)* $\frac{1}{(pq)^2}$ *e)* $\frac{1}{(5xy)^3} = \frac{1}{125x^3y^3}$

f) $2yz^5$ *g)* $\frac{1}{32}$ *h)* $-\frac{1}{8}$ *i)* $-\frac{1}{9}$ *j)* $\frac{2}{25}$

Exercise 3

Simplify the following:

a). $2a^3b^2 \times a^{-1} \times b^3$ b). $(5x^{-2}y)^{-3}$ c). $(3x^3y^{-1})^5$

d). $(a^{-4}b^{-5})^{-2}$ e). $\frac{a^2b^3c^{-4}}{a^4bc^5}$ f). $\frac{a^7 \times a^8 \times a^3}{a^2 \times a^5}$

g). $x\left(x - x^{-1}\right)$ h). $\frac{(2^4)^n}{2^3}$ i). $\frac{15a^2b}{3a^4b} \times \frac{4a^5b^2}{5a^3b^4}$

j). $2^4 - 2^3$

Answers to Exercise 3

a) $2a^2b^5$ *b*) $\frac{x^6}{5^3y^3}$ *c*) $\frac{3^5x^{15}}{y^5}$ *d*) a^8b^{10} *e*) $\frac{b^2}{a^2c^9}$

f) a^{11} *g*) $x^2 - 1$ *h*) 2^{4n-3} *i*) $\frac{4a^7b^3}{a^7b^5} = 4b^{-2} = \frac{4}{b^2}$ *j*) $2^3(2^1 - 2^0) = 2^3 = 8$