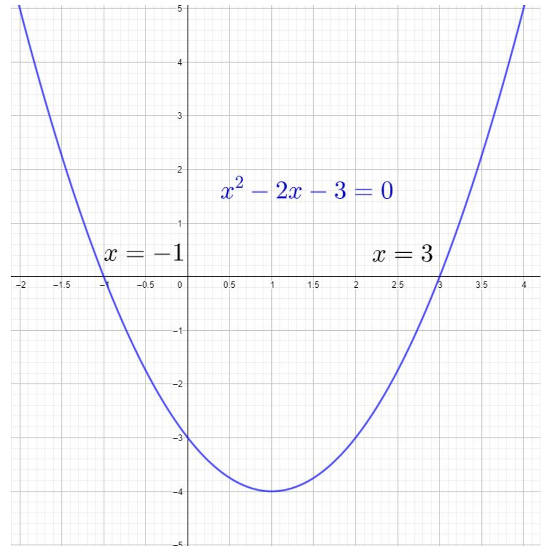
ES4 Quadratic Formula



The solutions to any quadratic equation $ax^2 + bx + c = 0$ can be found by substituting the values a, b, c into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solutions may be real or complex numbers.

In this module we only consider real solutions.

Example 1

Solve the equation:

$$x^2 - 5x + 4 = 0.$$

Solution:

We have[1] $a = 1,\ b = -5$ and $c = 4$. Substituting into the formula we get:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{5 \pm \sqrt{25 - 16}}{2} \\ &= \frac{5 \pm \sqrt{9}}{2} \\ &= \frac{5 \pm 3}{2} \\ &= \frac{5+3}{2} \text{ or } \frac{5-3}{2} \\ &= \frac{8}{2} \text{ or } \frac{2}{2} \\ &= 4 \text{ or } 1. \end{aligned}$$

The solution is $x = 1$ or $x = 4$.

[1] Note that $x^2 = 1 \times x^2$ and so $a = 1$.

Example 2

Solve the equation:

$$x^2 + x + 10 = 0.$$

Solution:

We have $a = 1$, $b = 1$ and $c = 10$. Substituting into the formula we get:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4(1)(10)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 - 40}}{2} \\ &= \frac{-1 \pm \sqrt{-39}}{2}. \end{aligned}$$

Since the $\sqrt{-39}$ does not exist[2] the equation has no solution.

[2] There is a solution if we use complex numbers but we are looking for real solutions.

Discriminant

The part of the quadratic formula which is under the radical sign ($b^2 - 4ac$) is called the discriminant. Its value determines the number of solutions and whether they will be rational or irrational.

- If $b^2 - 4ac < 0$ there are no (real) solutions. This means the graph of $y = ax^2 + bx + c$ doesn't touch or cut the x-axis.

- If $b^2 - 4ac = 0$ there is one solution. This means the graph of $y = ax^2 + bx + c$ touches the x-axis at this value of x.

- If $b^2 - 4ac > 0$ there are two solutions. This means the graph of $y = ax^2 + bx + c$ cuts the x-axis at these values of x.

- If $b^2 - 4ac$ is a perfect square[3] the solutions will be rational[4].

[3] A number is a perfect square if its square root is an integer. Integers are whole numbers: $\ldots, -2, -1, 0, 1, 2, \ldots$

[4] A rational number can be expressed as a fraction p/q where p and $q \neq 0$ are integers.

Exercise

Solve the following quadratic equations for real solutions (if possible):

1. $x - 4x - 7 = 0$
2. $x^2 - 2x - 2 = 0$
3. $x^2 + 6x - 9 = 0$
4. $x^2 - x - 7 = 0$
5. $4x^2 - 12x + 6 = 0$
6. $2x^2 + x - 2 = 0$
7. $x^2 - 4x + 2 = 0$
8. $2x^2 - 3x + 2 = 0$
9. $3x^2 + 5x - 7 = 0$
10. $3x^2 = x + 1$
11. $4x^2 + x + 3 = 0$
12. $\frac{3x+1}{2} = \frac{x+1}{x}$

Answers

1. $x = \frac{4 \pm \sqrt{44}}{2} = 2 \pm \sqrt{11}$
2. $x = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$
3. $x = \frac{-6 \pm \sqrt{72}}{2} = -3 \pm 3\sqrt{2}$
4. $x = \frac{1 \pm \sqrt{29}}{2}$
5. $x = \frac{12 \pm \sqrt{48}}{8} = \frac{3 \pm \sqrt{3}}{2}$
6. $x = \frac{-1 \pm \sqrt{17}}{4}$
7. $x = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$
8. *no solution*
9. $x = \frac{-5 \pm \sqrt{109}}{6}$
10. $x = \frac{1 \pm \sqrt{13}}{6}$
11. *no solution*
12. $x = -\frac{2}{3}, \quad x = 1$