DE4 SECOND ORDER NON-HOMOGENEOUS

Second Order Differential Equation with Constant Coefficients

The general expression of a second order differential equation is: $a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x)$

We shall only look at DE's where a_1, a_2, and a_3 are constants.

The Particular Integral

A reminder that the general form of a second order linear differential equation is:

$$a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x).$$

For second order differential equations where the function $f(x) \neq 0$, the general solution is:

$y(x) = y_c + y_p$ (complimentary function + particular integral)

First determine the complimentary function (as for homogeneous second order DEs), and then generate the particular integral.

It is usual for the particular integral to be a type of function that is based on the general form of the function $f(x)$.

Here are some examples:

If $f(x) = 4$ let $y_p = a$

If $f(x) = 2x + 3$ let $y_p = ax + b$

If $f(x) = x^2 - 2x + 7$ let $y_p = ax^2 + bx + c$

If $f(x) = 2\,e^{\alpha x}$ let $y_p = a\,e^{\alpha x}$

If $f(x) = 3 \sin px$ OR $7 \cos px$ OR $4 \sin px + 2 \cos px$ let $y_p = a \sin px + b \cos px$

Example

Determine the general solution $y(x)$ for the equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3x + 2$

Solution:

Auxiliary equation: $2m^2 - m - 6 = 0$

Factorise: $(2m + 3)(m - 2) = 0$

$\therefore m = -\frac{3}{2}$ or $m = 2$ Two real and different solutions.

Complimentary function: $y_c = Ae^{-\frac{3}{2}x} + Be^{2x}$

Particular integral: $f(x) = \text{linear function}$

General form of a linear function is $y = ax + b$

$\therefore$ Let $y_p = ax + b$

$y_p' = a$

$y_p'' = 0$

Substitute for $y_p,\ y_p',$ and y_p'' into the original DE.

$$2(0) - a - 6(ax + b) = 3x + 2$$

Grouping *'like'* powers of *x*:

$$(-6a)x + (-a - 6b) = 3x + 2$$

Equate coefficients of powers of x to solve for a and b.

$x^1: \ -6a = 3 \qquad \therefore a = -\frac{1}{2}$

$x^0: \ -a - 6b = 2 \qquad \therefore b = -\frac{1}{4}$

$\therefore$ The particular integral $y_p = -\frac{1}{2}x - \frac{1}{4}$

The general solution, $y(x) = y_c + y_p$

$y(x) = Ae^{-\frac{3}{2}x} + Be^{2x} - \frac{1}{2}x - \frac{1}{4}$

Repeated solution

In some instances a term in the complimentary function will be repeated in the particular integral, in which case the particular integral solution will not 'work'.

When this occurs, multiply the initial particular integral by successive powers of x until it is no longer contained within the complimentary function.

Example

Solve the differential equation

$y'' - y' - 2y = 6e^{-x}$

Sol<u>n:</u>

Auxiliary equation: $m^2 - m - 2 = 0$

Factorise: $(m + 1)(m - 2) = 0$

$\therefore m = -1$ or $m = 2$ Two real and different solutions.

Complimentary function: $y_c = Ae^{-x} + Be^{2x}$

Particular integral: $f(x) = 6e^{-x}$

Let: $y_p = ke^{-x}$.

However this term is already contained in the complimentary function.

$\therefore$ Let $y_p = kxe^{-x}$

$y_p' = ke^{-x} - kxe^{-x}$ using the product rule for differentiation

$y_p'' = -2ke^{-x} + kxe^{-x}$

Substitute for y_p, y_p', and y_p'' into the original DE.

$-2ke^{-x} + kxe^{-x} - (ke^{-x} - kxe^{-x}) - 2(kxe^{-x}) = 6e^{-x}$

$-3ke^{-x} = 6e^{-x}$

$k = -2$

$\therefore$ The particular integral $y_p = -2xe^{-x}$

The general solution, $y(x) = y_c + y_p$

$$y(x) = Ae^{-x} + Be^{2x} - 2xe^{-x}$$

Exercise

a. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 6$ Given $y(0) = 0$ and $\frac{dy}{dx}(0) = 0$

b. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 5e^x$

c. $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = 4\sin 2x$

d. $\frac{d^2y}{dx^2} + 9y = 12\cos 3x$ Given $y(0) = 2$ and $\frac{dy}{dx}(0) = 3$

e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4x + 3\cos 2x$

f. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 6e^x \sin 2x$

g. When a mass of 2 kilograms is attached to a spring whose constant is 32 Newtons per metre it comes to rest in the equilibrium position. Starting at $t = 0$ a force equal to $f(t) = 68e^{-2t}\cos 4t$ is applied to the system. Find the equation of motion in the absence of damping.

h. A mass of 40 g stretches a spring 10 cm. A damping device imparts a resistance to motion equal to 560 times the instantaneous velocity. Find the equation of motion if the mass is released from the equilibrium position with a downward velocity of 2 cm/sec.

Answers

a. $y = \frac{3}{8}e^{-4x} + \frac{3}{2}x - \frac{3}{8}$

b. $y = e^x(Ax + B + \frac{5}{2}x^2)$

c. $y = Ae^{\frac{1}{2}x} + Be^{-3x} - \frac{4}{221}(11\sin 2x + 10\cos 2x)$

d. $y = 2\cos 3x + (1 + 2x)\sin 3x$

e. $y = (Ax + B)e^{2x} + x + 1 - \frac{3}{8}\sin 2x$

f. $y = e^{-x}(A\cos x + B\sin x) + \frac{e^x}{17}(6\sin 2x - 24\cos 2x)$

g. $x(t) = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t + \frac{1}{2}e^{-2t}\cos 4t - 2e^{-2t}\sin 4t$

h. $x(t) = \frac{2}{7}e^{-7t}\sin 7t$