DE1 FIRST ORDER SEPARABLE

A first order variables separable differential equation is one in which the variables can be separated into either side of the equation.

Example 1

Solve for $y(x)$ given $\frac{dy}{dx} - x^2y = 0$

$\frac{dy}{dx} = x^2y$

$\frac{1}{y}\frac{dy}{dx} = x^2$

$\frac{1}{y}\,dy = x^2dx$

$\int \frac{1}{y}\,dy = \int x^2dx$

$\ln y = \frac{x^3}{3} + c$

$e^{\ln y} = e^{\frac{x^3}{3}+c}$

$y = e^{\frac{x^3}{3}}.e^c$ (let $A = e^c$)

$y = Ae^{\frac{x^3}{3}}$

Example 2

Solve for $y(x)$ given $\frac{dy}{dx} = 3y + 1$ $y(0) = 1$

Soln:

$\frac{dy}{dx} = 3y + 1$

$\frac{1}{3y+1}\frac{dy}{dx} = 1$

$\int \frac{1}{3y+1}\,dy = \int 1\,dx$

$\int \frac{1}{3y+1}\,dy$ Solve this integral using method of substitution

Let $u = 3y + 1 \rightarrow \frac{du}{dy} = 3 \rightarrow dy = \frac{du}{3}$

$\therefore \int \frac{1}{u}\frac{du}{3} = \int 1\,dx$

$\frac{1}{3}\ln(u) = x + c_1$

$\frac{1}{3}\ln(3y + 1) = x + c_1$

$\ln(3y + 1) = 3(x + c_1)$

$3y + 1 = e^{3x+3c_1}$

$$y = \frac{1}{3}(e^{3x+3c_1} - 1)$$

$$y = \frac{1}{3}\, e^{3x} e^{3c_1} - \frac{1}{3} \qquad \text{Let } A = \frac{1}{3} e^{3c_1}$$

$$y = A\, e^{3x} - \frac{1}{3}$$

Substitute given boundary values to solve for *A*.

$$\therefore\ 1 = A\, e^{3\times 0} - \frac{1}{3}$$

$$A = \frac{4}{3}$$

$$\therefore y(x) = \frac{4}{3}\ e^{3x} - \frac{1}{3}$$

Exercise

a. $\frac{dy}{dx} = \sin 5x$
b. $dx + e^{3x} dy = 0$
c. $(x+1)\frac{dy}{dx} = x + 6$
d. $xy' = 4y$
e. $\frac{dy}{dx} = \frac{y^3}{x^2}$
f. $\frac{dy}{dx} = \frac{x^2 y^2}{1+x}$
g. $\frac{dy}{dx} = e^{3x+2y}$
h. $\frac{dy}{dx} = y\ x\ sin5x\ ; \quad y(0) = 1$
i. $\frac{dy}{dt} + ty = y\ ; \quad y(1) = 3$
j. $\frac{dy}{dx} = y^2 - 1\ ; \quad y(0) = 2$

Answers

a. $y = -\frac{1}{5} cos5x + c$
b. $y = \frac{1}{3} e^{-3x} + c$
c. $y = x + 5ln|x+1| + c$
d. $y = cx^4$
e. $y = \pm\sqrt{\frac{x}{Cx+2}}$
f. $y = \left[-\frac{1}{2}x^2 + x - ln|x+1| + C\right]^{-1}$
g. $y = -\frac{1}{2}\ln(\frac{2}{3}e^{3x} + C)$
h. $y = e^{-\frac{x}{5}cos5x + \frac{1}{25}sin5x}$
i. $y = 3e^{t - \frac{1}{2}t^2 - \frac{1}{2}}$
j. $y = \frac{3+e^{2x}}{3-e^{2x}}$.