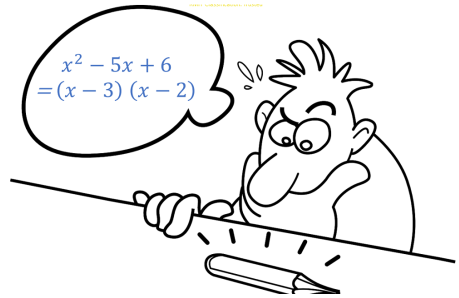
A3.4 Factorisation: Quadratics



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The general form of a quadratic expression is: $ax^2 + bx + c$, $a \neq 0$, where a, b and c are real constants and x is the variable.

We will initially work with expressions that have $a = 1$ so the expression becomes $x^2 + bx + c$.

Expansion

To expand an expression of the form $(a+b)(c+d)$, multiply each term in the first bracket by each term in the second bracket.

$$\begin{aligned}(x+2)(x+3) &= x(x+3) + 2(x+3) \\ &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6.\end{aligned}$$

Factorisation

Factorisation is the reverse of expansion:

$x^2 + 5x + 6$ is expressed as the product of two factors, $(x+2)$ and $(x+3)$. That is

$$x^2 + 5x + 6 = (x+2)\,(x+3)\,.$$

Note that:

- Multiplying the first term in each bracket gives the term x^2 in the expression as above.
- Multiplying the last term in each bracket gives the constant term, $+6$ in the expression.
- The coefficient of the x term is the sum of the last term in each bracket $(+2+3 = +5)$.

Watch a short video on factorising quadratics

Download transcription of video on Factorising quadratics

The basic rule is:

To factorise $x^2 + bx + c$, find two numbers m and n such that

$$x^2 + bx + c = (x + m)(x + n)$$

where $m \times n = c$ and $m + n = b$.

Note that order of the factors does not matter. That is

$$\begin{aligned} x^2 + bx + c &= (x + m)(x + n) \\ &= (x + n)(x + m). \end{aligned}$$

Example 1

Factorise $x^2 + 9x + 14$.

Solution:

We want to write

$$x^2 + 9x + 14 = (x + m)(x + n)$$

where according to the rule above,

$$\begin{aligned} m \times n &= 14 \text{ and} \\ m + n &= 9. \end{aligned}$$

The factors of 14 are

1. $m = 1$, $n = 14$	2. $m = -1$, $n = -14$	3. $m = 2$, $n = 7$	4. $m = -2$, $n = -7$.

Of these, only the factors in 3 satisfy the requirement that $m + n = 9$. So

$$x^2 + 9x + 14 = (x + 2)(x + 7).$$

Example 2

Factorise $y^2 - 7y + 12$.

Solution:

We want to write

$$y^2 - 7x + 12 = (y + m)(y + n)$$

where according to the rule above,

$$\begin{aligned} m \times n &= 12 \text{ and} \\ m + n &= -7. \end{aligned}$$

The factors of 12 are

$$\begin{array}{llllll} 1.\ m=3 & 2.\ m=-3 & 3.\ m=2 & 4.\ m=-2 & 5.\ m=12 & 6.\ m=-12 \\ \quad n=4 & \quad n=-4 & \quad n=6 & \quad n=-6 & \quad n=1 & \quad n=-1. \end{array}$$

Of these, only the factors in 2 satisfy the requirement that $m + n = -7$. So

$$y^2 - 7x + 12 = (y-3)(y-4).$$

Example 3

Factorise $p^2 - 5p - 14$.

Solution:

We want to write

$$p^2 - 5p - 14 = (p+m)(p+n)$$

where according to the rule above,

$$m \times n = 14 \text{ and}$$
$$m + n = -5.$$

The factors of -14 are

$$\begin{array}{llll} 1.\ m=1 & 2.\ m=-1 & 3.\ m=-2 & 4.\ m=2 \\ \quad n=-14 & \quad n=14 & \quad n=7 & \quad n=-7. \end{array}$$

Of these, only the factors in 4 satisfy the requirement that $m + n = -5$. So

$$p^2 - 5x - 14 = (p+2)(p-7).$$

Example 4

Factorise $a^2 + 6a - 7$.

Solution:

We want to write

$$a^2 + 6a - 7 = (a+m)(a+n)$$

where according to the rule above,

$$m \times n = -7 \text{ and}$$
$$m + n = 6.$$

The factors of -7 are

1. $m = 1$, $n = -7$
2. $m = -1$, $n = 7$

Of these, only the factors in 2 satisfy the requirement that $m + n = 6$. So

$$a^2 + 6a - 7 = (a - 1)(a + 7).$$

Example 5 (No Real Factors)

Factorise $a^2 + 3a + 6$.

Solution:

We want to write

$$a^2 + 3a + 6 = (a + m)(a + n)$$

where according to the rule above,

$$m \times n = 6 \text{ and}$$
$$m + n = 3.$$

The factors of 6 are

1. $m = 1$, $n = 6$
2. $m = -1$, $n = -6$
3. $m = -2$, $n = -3$
4. $m = 2$, $n = 3$.

None of these factors satisfy the requirement that $m + n = 3$. So it is not possible to factorise the expression

$$a^2 + 3a + 6.$$

In this case we say there are no real factors.[1]It is important to understand when this occurs and is discussed in a later section.

[1] Geometrically this means that the graph of $y = a^2 + 3a + 6$ does not touch or intersect the $a -$ axis. There are complex factors but these are not dealt with in this module.

Factorisation when $a \neq 1$.

In this section we deal with factorisation of expressions of the form

$$ax^2 + bx + c$$

where $a \neq 1$.

Expressions of the type $ax^2 + bx + c$ can be factorised using a technique similar to that used for expressions of the type $x^2 + bx + c$.

In this case the coefficient of x, in at least one bracket, will not equal 1.

Consider the following product.

$$
\begin{aligned}
(3x+2)(2x+1) &= (3x)(2x)+(3x)(1)+(2)(2x)+(2)(1) \\
&= 6x^2+3x+4x+2 \\
&= 6x^2+7x+2.
\end{aligned}
$$

Note that

- multiplying the first term in each bracket gives the x^2 term. In this case $6x^2$.
- multiplying the last term in each bracket gives the constant term, in this case 2.
- the coefficient of the $x-$term is the sum of the x terms in the expansion. In this case $3x+4x=7x$.
  We can use these ideas to factorise expressions like ax^2+bx+c.

Example 6

Factorise $2x^2+7x+6$.

Solution:

The only factors of the coefficient of the x^2 term are 2 and 1. So we are looking for a factorisation like

$$
\begin{aligned}
2x^2+7x+6 &= (2x+m)\,(x+n) \\
&= 2x^2+2nx+mx+nm \\
&= 2x^2+(2n+m)\,x+nm
\end{aligned}
$$

where $mn=6$ and $2n+m=7$. We have the following possibilities:[2]

[2] Note that negative factors like $m=-6$, $n=-1$ and $m=-1$, $n=-6$ don't need to be considered as they don't satisfy condition $2n+m=7$.

1. $m=3, \;\; n=2$
2. $m=2, \;\; n=3$
3. $m=6, \;\; n=1$

Of these possibilities, the only one that satisfies $2n+m=7$ is number 1. That is $m=2$ and $n=2$, so

$$2x^2+7x+6=(2x+3)\,(x+2)\,.$$

Example 7

Factorise $2x^2-10x+12$.

Solution:

At first this looks like a case where $a=2$ but a factor of 2 can be taken out to get: [3]

[3] You should always check if there is a number that divides into all terms of the quadratic.

$$2x^2 - 10x + 12 = 2\left(x^2 - 5x + 6\right)$$

Now we can use the methods in Examples $1-4$ above to get

$$2x^2 - 10x + 12 = 2\,(x+m)\,(x+n)$$

where $mn = 6$ and $m + n = -5$. This implies $m = -2$ and $n = -3$ and so:

$$\begin{aligned}2x^2 - 10x + 12 &= 2\left(x^2 - 5x + 6\right)\\ &= 2\,(x-2)\,(x-3)\,.\end{aligned}$$

Example 8

Factorise $6x^2 + 13x - 8$.

Solution:

In this case there are no numbers that divide into each term as we had in Example 7 above. The coefficient of the x^2 term is 6 which has factors

$$\begin{aligned}6 &= 6 \times 1\\ &= 2 \times 3.\end{aligned}$$

So we are looking for a factorisation such as:

$$\begin{aligned}6x^2 + 13x - 8 &= (6x+m)\,(x+n)\\ &= 6x^2 + 6xn + mx - 8\\ &= 6x^2 + (6n+m)\,x - 8 \qquad (8.1)\end{aligned}$$

or

$$\begin{aligned}6x^2 + 13x - 8 &= (3x+m)\,(2x+n)\\ &= 6x^2 + 3xn + 2mx - 8\\ &= 6x^2 + (3n+2m)\,x - 8 \qquad (8.2)\end{aligned}$$

In both cases, $mn = -8$. So we have the possibilities

$$\begin{aligned}m &= -8 & n &= 1\\ m &= 8 & n &= -1\\ m &= 4 & n &= -2\\ m &= -4 & n &= 2.\end{aligned}$$

For eqn (8.1) we know that $6n + m = 13$. This is not satisfied by any of the m and n values above.

For eqn (8.2) we know that $3n + 2m = 13$. This is satisfied by $m = 8$ and $n = -1$ and so

$$6x^2 + 13x - 8 = (3x+8)\,(2x-1)\,.$$

Example 9

Factorise $4x^2 + 4x + 1$

Solution:

In this case there are no numbers that divide into each term as we had in Example 7 above. The coefficient of the x^2 term is 4 which has factors

$$\begin{aligned} 4 &= 4 \times 1 \\ &= 2 \times 2. \end{aligned}$$

So we are looking for a factorisation such as:

$$\begin{aligned} 4x^2 + 4x + 1 &= (4x + m)(x + n) \\ &= 4x^2 + 4xn + mx + mn \\ &= 4x^2 + (4n + m)x + 1 \qquad (9.1) \end{aligned}$$

or

$$\begin{aligned} 4x^2 + 4x + 1 &= (2x + m)(2x + n) \\ &= 4x^2 + 2xn + 2mx + mn \\ &= 4x^2 + (2n + 2m)x + 1 \qquad (9.2) \end{aligned}$$

where $mn = 1$. That means

$$\begin{aligned} m &= 1 & n &= 1 \qquad (9.3) \\ m &= -1 & n &= -1. \qquad (9.4) \end{aligned}$$

Suppose eqn (9.1) is correct then $(4n + m) = 4$. But this is not possible with the choices for m and n in (9.3) and (9.4). Hence the factorisation must be as in eqn (9.2) with $2n + 2m = 4$. The latter is achieved with (9.3) and so

$$4x^2 + 4x + 1 = (2x + 1)(2x + 1).$$

The approach given in examples 6-9 is okay provided there are not too many factors for a and c. If the number of factors is excessive, we can employ other methods.

When Can you Get Real Linear Factors for a Quadratic?

In Example 5 above we found that we could not get real linear factors for

$$a^2 + 3a + 6.$$

This raises the question of when real solutions to general quadratics may be found. The most general quadratic has the form [4]

[4] Note that the graph of

$$y = ax^2 + bx + c$$

is a parabola.

$$ax^2 + bx + c.$$

To determine if there are real linear factors, we introduce the discriminant.

The Discriminant

The discriminant denoted Δ, for the general quadratic

$$ax^2 + bx + c$$

is

$$\Delta = b^2 - 4ac.$$

If

$$\Delta = \begin{cases} 0 \text{ there is one repeated linear factor} \\ > 0 \text{ there are two distinct linear factors} \\ < 0 \text{ there are no real linear factors.} \end{cases}$$

The discriminant tells us how many real roots there are to the equation [5]

$$ax^2 + bx + c = 0.$$

[5] Geometrically, the discriminant tells us how many times the graph of

$$y = ax^2 + bx + c$$

intersects the $x-$axis. If $\Delta = 0$, the graph just touches the $x-$axis at one point. If $\Delta > 0$, the graph intersects the $x-$axis at two points. If $\Delta < 0$, the graph does not intersect the $x-$axis.

Example 10

Factorise (if possible) $x^2 + 6x + 12$

Solution:

In this case: $a = 1$, $b = +6$, $c = +12$ therefore

$$\begin{aligned} \Delta &= \left(b^2 - 4ac\right) \\ &= 36 - 4(1)(12) \\ &= 36 - 48 \\ &= -12 \\ &< 0. \end{aligned}$$

The discriminant is negative therefore $x^2 + 6x + 12$ has no real factors.

Exercise 1

Factorise the following expressions (if possible):

a) $x^2 + 10x + 21$ b) $z^2 + 11z + 18$ c) $x^2 + 5x - 14$
d) $m^2 - m - 72$ e) $x^2 + 6x + 9$ f) $a^2 - 15a + 44$
g) $x^2 - 2x - 24$ h) $y^2 - 10y + 16$ i) $z^2 + 4z - 60$
j) $n^2 + 6n - 16$ k) $a^2 + 5a + 10$ l) $s^2 + 2s - 48$
m) $y^2 + 7y + 19$ n) $x^2 + 16x + 39$ o) $x^2 - 14x + 45$

Answers

Note that order of factors does not matter.

a) $(x+7)(x+3)$ b) $(z+9)(z+2)$ c) $(x+7)(x-2)$
d) $(m-9)(m+8)$ e) $(x+3)(x+3)$ f) $(a-11)(a-4)$
g) $(x-6)(x+4)$ h) $(y-2)(y-8)$ i) $(z-6)(z+10)$
j) $(n-2)(n+8)$ k) no real factors l) $(s+8)(s-6)$
m) no real factors n) $(x+13)(x+3)$ o) $(x-9)(x-5)$.

Exercise 2

Factorise the following if possible.

a) $5x^2+13x+6$ b) $2x^2+x-15$ c) $3m^2-m-2$
d) $3y^2-10y+8$ e) $2a^2+11a+12$ f) $6x^2-11x+5$

Answers

Note that order of factors does not matter.

a) $(5x+3)(x+2)$ b) $(2x-5)(x+3)$ c) $(3m+2)(m-1)$
d) $(3y-4)(y-2)$ e) $(2a+3)(a+4)$ f) $(6x-5)(x-1)$.