A3.2 Factorisation: Perfect Squares

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The idea of a perfect square permeates mathematics and especially algebra. Properties of perfect squares will help you in factorising and expanding algebraic expressions. An understanding of perfect squares is essential for success in mathematics.

Introduction

Examples of perfect squares are 5^2, x^2, a^2b^2, $(xy)^2$ and $(a \pm b)^2$. For your development of algebraic skills, the most interesting of these are $(a+b)^2$ and $(a-b)^2$.

Rules for Expanding Perfect Squares

Consider $(a+b)^2$. We have

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \text{ by definition}\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2. \qquad (1)\end{aligned}$$

Similarly,

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b) \text{ by definition}\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2. \qquad (2)\end{aligned}$$

You must remember the equations (1) and (2). That is

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Note that

- the first and last terms must be positive and must be perfect squares.
- the middle term must be twice the product of the first and last terms and may be positive or negative.

Using Perfect Squares for Factorising

The rules for expanding perfect squares may be used in reverse to factorise algebraic expressions.

Example 1.

Factorise $x^2 + 14x + 49$.
Solution:
Note that $49 = 7^2$ and $14 = 2 \times 7$. Applying equation (1)above,

$$x^2 + 14x + 49 = (x + 7)^2 .$$

And $x^2 + 14x + 49$ is a perfect square.

Example 2.

Is $y^2 - 20y + 25$ a perfect square?
Solution:
Note that $25 = 5^2$ but $-20 \neq 2 \times 5$ hence $y^2 - 20y + 25$ is not a perfect square.

Example 3.

Is $4a^2 - 12a - 9$ a perfect square?
Solution:
The last term is negative and so $4a^2 - 12a - 9$ is not a perfect square.

Example 4.

Is $100x^2 - 180x + 81$ a perfect square?
Solution:

$$\begin{aligned} 100x^2 &= (10x)^2 \\ 81 &= 9^2 \end{aligned}$$

and $180 = 2 \times 10 \times 9$. Hence

$$100x^2 - 180x + 81 = (10x - 9)^2$$

and is a perfect square.

Example 5.

Factorise $50x^2 + 80x + 32$?

Solution:

At first sight, the expression is not a perfect square because neither $50x^2$ nor 32 is a perfect square. However, if we divide the expression by 2 we have

$$50x^2 + 80x + 32 = 2\left(25x^2 + 40x + 16\right) \tag{3}$$

The expression $25x^2 + 40x + 16$ is a perfect square because

$$25x^2 + 40x + 16 = (5x + 4)^2$$

and so from equation (3)

$$50x^2 + 80x + 32 = 2\,(5x + 4)^2\,.$$

The answer is that $50x^2 + 80x + 32$ is not a perfect square. However $50x^2 + 80x + 32 = 2\,(5x + 4)^2$ which is a useful factorisation.

Exercise

Check each of the following expressions. If it is a perfect square, state the perfect square.

1. $a^2 + 2a + 1$
2. $x^2 - 4x + 4$
3. $25x^2 - 10x + 1$
4. $4y^2 - 6y + 9$
5. $81x^2 + 108x + 36$
6. $9a^2 - 24a - 16$
7. $16x^2 - 40xy + 25y^2$
8. $121z^2 + 88z + 64$
9. $2x^2 + 8x + 8$.

Answers

1. $(a + 1)^2$
2. $(x - 2)^2$
3. $(5x - 1)^2$
4. Not a perfect square
5. $(9x + 6)^2$
6. Not a perfect square
7. $(4x - 5y)^2$
8. Not a perfect square
9. Not a perfect square but $2x^2 + 8x + 8 = 2\,(x + 2)^2$