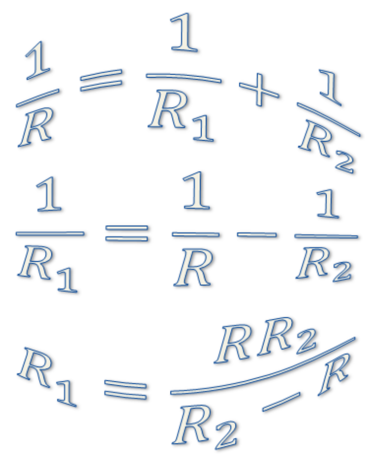
A2.3 Transposition of Formulas with Challenges



In this tip sheet we look at some more complicated formulas where the new subject variable:

- appears more than once in the formula, like m in $E = mgh + \frac{1}{2}mv^2$.
- is on the bottom (the denominator) of a fraction, like v in $1/f = 1/v - 1/u$.
- is an exponent, like t in $Q = Ae^{kt}$.

Subject Variable Appears More Than Once in the Formula

The strategy is to move all terms containing the desired subject variable to the same side of the equal sign and factorize.

Examples

1. Make m the subject[1] in the formula:

$$E = mgh + \frac{1}{2}mv^2.$$

In this case the m 's are already on the same side of the equation so we can take out m as a factor:

$$E = m\left(gh + \frac{1}{2}v^2\right).$$

To get m on its own we divide both sides by $gh + \frac{1}{2}v^2$ to get:

$$\frac{E}{gh + \frac{1}{2}v^2} = \frac{m\left(gh + \frac{1}{2}v^2\right)}{gh + \frac{1}{2}v^2}$$
$$= m \quad \text{(after cancelling)}.$$

[1] In the formula, E is the subject because we have $E = \cdots$. We want to rearrange to get $m = \cdots$.

That is,

$$m = \frac{E}{gh + \frac{1}{2}v^2}.$$

It is possible to improve the look of the answer by getting rid of the $1/2$ in the denominator. We do this by multiplying the top and bottom of the right hand side by 2 to get:

$$\begin{aligned} m &= \frac{E}{gh + \frac{1}{2}v^2} \times \frac{2}{2} \\ &= \frac{2E}{2gh + v^2}. \end{aligned}$$

2. Make I the subject of the formula $Ir = E - IR$.
   First get all terms involving I to one side by adding IR to both sides:

$$\begin{aligned} Ir + IR &= E - IR + IR \\ &= E. \end{aligned}$$

   Take I out as common factor:

$$I\left(r + R\right) = E.$$

   Now divide both sides by $r + R$:

$$I = \frac{E}{r + R}.$$

The examples above showed all steps in detail. You don't have to do that when answering questions. For example 1 above you would write:

$$\begin{aligned} E &= mgh + \frac{1}{2}mv^2 \\ &= m\left(gh + \frac{1}{2}v^2\right) \\ m &= \frac{E}{gh + \frac{1}{2}v^2}. \end{aligned}$$

For example 2 above you would write:

$$\begin{aligned} Ir &= E - IR \\ Ir + IR &= E \\ I\left(r + R\right) &= E \\ I &= \frac{E}{r + R}. \end{aligned}$$

Subject Variable is a Fraction

More particularly, we mean the desired subject is in the denominator[2] of a fraction.

[2] The numerator is the number or expression on the top of a fraction and the denominator is the number or expression on the bottom.

> The strategy is to get the desired subject on the top line by multiplying by a common divisor of the denominators. You then proceed as usual.

Examples

1. For example, make u the subject of the formula $1/f = 1/v - 1/u$. A suitable common divisor of the denominators[3] is fvu. Multiplying both sides of the formula by fvu gives

$$\frac{1}{f}fvu = \frac{1}{v}fvu - \frac{1}{u}fvu$$
$$vu = fu - fv \quad \text{(after canceling the terms)}.$$

Next step is to get all terms involving u on the same side:

$$vu - fu = -fv \quad \text{(subtracting } fu \text{ from both sides)}.$$

Now we factorize the left hand side:

$$u\,(v - f) = -fv.$$

The final step is to divide both sides by $v - f$ to get

$$u = \frac{-fv}{v - f} = -\frac{fv}{v - f}.$$

This is a correct answer but having a $-$ sign in front is ugly. We get rid of it by multiplying the top and bottom of the right hand side by -1:

$$u = \frac{-fv\,(-1)}{(v - f)\,(-1)} = \frac{fv}{(f - v)}.$$

The example above showed all steps. You don't have to provide

[3] A common divisor is the product of the denominators. For example to simplify $\frac{1}{4} + \frac{1}{8}$ you can use a lowest common denominator of 8 or a common denominator of $8 \times 4 = 32$. Either will give the same result, though the lowest common denominator involves smaller numbers. To simplify algebraic fractions like $\frac{1}{v} - \frac{1}{u}$ we must use a common denominator $v \times u = vu$ as we don't know the values of v and u.

such detail. An acceptable solution would be:

$$\begin{aligned}\frac{1}{f} &= \frac{1}{v} - \frac{1}{u} \\ vu &= fu - fv \\ u\left(v - f\right) &= -fv \\ u &= \frac{-fv}{v - f} \\ &= \frac{fv}{f - v}.\end{aligned}$$

2. Transpose $m = \sqrt{\frac{d-s}{s(e-f)}}$ to make s the subject.
The square root sign acts as a bracket[4] and we want to remove it so we can get at the subject term s. We remove it by squaring both sides:

[4] The term $\frac{d-s}{s(e-f)}$ is considered as a single term.

$$\begin{aligned}m^2 &= \left(\sqrt{\frac{d-s}{s\left(e-f\right)}}\right)^2 \\ &= \frac{d-s}{s\left(e-f\right)}.\end{aligned}$$

Now multiply both sides by $s\left(e-f\right)$[5] to get:

[5] Note that $m^2 = m^2/1$ and so the common denominator of both sides is $1 \times s\left(e-f\right) = s\left(e-f\right)$. By multiplying by $s\left(e-f\right)$we remove fractions.

$$s\left(e-f\right)m^2 = d - s.$$

Now we gather all the s terms to one side. We add s to both sides:

$$s\left(e-f\right)m^2 + s = d.$$

Take a common factor of s out of the right hand side:

$$s\Big(\left(e-f\right)m^2 + 1\Big) = d.$$

Now we divide by $\big(\left(e-f\right)m^2 + 1\big)$ to get the result

$$s = \frac{d}{\left(\left(e-f\right)m^2 + 1\right)}.$$

The example above showed all steps. You don't have to provide such detail. An acceptable solution would be

$$\begin{aligned}m &= \sqrt{\frac{d-s}{s\left(e-f\right)}} \\ m^2 &= \frac{d-s}{s\left(e-f\right)} \\ s\left(e-f\right)m^2 &= d - s \\ s\left(e-f\right)m^2 + s &= d \\ s\Big(\left(e-f\right)m^2 + 1\Big) &= d \\ s &= \frac{d}{\left(\left(e-f\right)m^2 + 1\right)}.\end{aligned}$$

Subject Variable is an Exponent

> The strategy is to use rules of logarithms to remove the desired subject from an index. The essential rule is:
> $\log a^n = n \log a$.
> The following log rules may then be required:
> $\log ab = \log a + \log b$
> $\log \frac{b}{a} = \log b - \log a$

Examples

1. Make t the subject in the formula $Q = Ae^{kt}$.
   Take $\log_e$of both sides to get

$$\log_e Q = \log_e \left(Ae^{kt} \right)$$

Use log laws[6] to simplify the right hand side:

$$\log_e Q = \log_e A + \log_e \left(e^{kt} \right)$$

Use log laws[7] to simplify the second term on the right hand side:[8]

$$\begin{aligned} \log_e Q &= \log_e A + kt \log_e e \\ &= \log_e A + kt \end{aligned}$$

Now rearrange for kt and use log laws:[9]

$$\begin{aligned} kt &= \log_e Q - \log_e A \\ &= \log_e \left(\frac{Q}{A} \right). \end{aligned}$$

Finally divide both sides by k to get[10]

$$t = \frac{1}{k} \log_e \left(\frac{Q}{A} \right).$$

This example shows all steps. You don't have to provide such detail. An acceptable solution would be

$$\begin{aligned} \log_e Q &= \log_e \left(Ae^{kt} \right) \\ &= \log_e A + kt \\ kt &= \log_e Q - \log_e A \\ &= \log_e \frac{Q}{A} \\ t &= \frac{1}{k} \log_e \frac{Q}{A} \end{aligned}$$

[6] $\log(ab) = \log_e a + \log_e b$.

[7] $\log x^n = n \log x$.

[8] Remember $\log_e e = 1$.

[9] $\log b - \log a = \log \left(\frac{b}{a} \right)$.

[10] Remember dividing by k is the same as multiplying by $1/k$.

2. Make n the subject of the formula $S = P(1+i)^n$. In this example, we show explanatory notes on each line. You don't have to provide these notes in your work.
   We have:

$$\begin{aligned}
S &= P(1+i)^n \\
\log S &= \log\left[P(1+i)^n\right] \quad \text{(taking logs of both sides)} \\
&= \log P + \log(1+i)^n \quad \text{(using log laws)} \\
&= \log P + n\log(1+i) \quad \text{(using log laws)} \\
n\log(1+i) &= \log S - \log P \quad \text{(rearranging)} \\
&= \log\left(\frac{S}{P}\right) \quad \text{(using log laws)} \\
n &= \frac{\log\left(\frac{S}{P}\right)}{\log(1+i)}.
\end{aligned}$$

3. Transform the formula $T = \frac{1}{c}\log_e(m-A)$ to make m the subject[11]. In this example, we show explanatory notes on each line. You don't have to provide these notes in your work.
   We have:

$$\begin{aligned}
T &= \frac{1}{c}\log_e(m-A) \\
cT &= \log_e(m-A) \quad \text{multiplying both sides by } c \\
e^{cT} &= e^{\log_e(m-A)} \quad \text{exponentiating both sides using base } e \\
e^{cT} &= m - A \quad \text{using log laws} \\
e^{cT} + A &= m \quad \text{adding } A \text{ to both sides} \\
m &= A + e^{cT} \quad \text{after rearranging.}
\end{aligned}$$

[11] In this example we use the fact that $cT = \log_e(m-A)$ is equivalent to $e^{cT} = m - A$.

Exercises

Transpose the following formulae to make the variable in the square brackets [ ] the subject.

1. $M = 10.5C + 35.2(W - c/8)$ $[C]$
2. $At = M(P+t)$ $[t]$
3. $I = E/R$ $[R]$
4. $\frac{P}{Q} = \frac{R}{S}$ $[S]$
5. $I = E/(R+r)$ $[r]$
6. $W = \frac{2PR}{R-r}$ $[R]$
7. $A = \sqrt{\frac{2q(L-r)}{rL}}$ $[L]$
8. $E = \frac{w^2 a}{(w^2+m)b^3}$ $[w]$
9. $H = Ae^{-kt}$ $[t]$
10. $\frac{1}{q^2} = \log_e(M/2) = P$ $[M]$

Answers

Note there can be algebraically equivalent answers that look different to the answers below.

1. $C = \frac{M-35.2W}{61}$
2. $t = \frac{MP}{A-M}$
3. $R = \frac{E}{I}$
4. $S = \frac{RQ}{P}$
5. $r = \frac{E-IR}{I}$
6. $R = \frac{wr}{w-2P}$
7. $L = \frac{2qr}{2q-A^2r}$
8. $w = \pm\sqrt{\frac{Eb^3m}{a-Eb^3}}$
9. $t = -\frac{1}{k}\log_e\left(\frac{H}{A}\right)$
10. $M = 2e^{pq^2}$