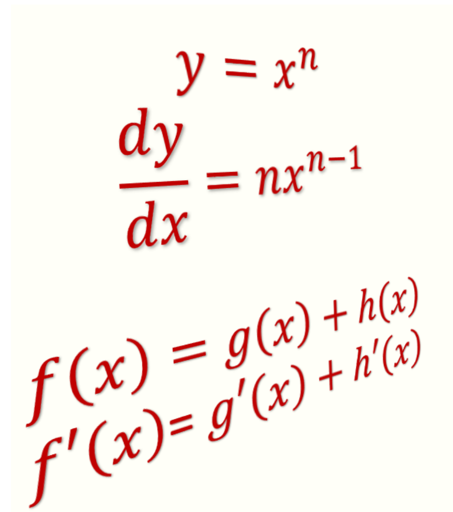
D4: Rules for Differentiation



It is not always convenient to use differentiation from first principles to find a derivative function. The "rules" shown below have been established from first principles and can be used to find derivative functions directly.

View short video on rules for differentiation.

Alternative Notations

We will also introduce alternative notations for derivatives. When y is a function of x, i.e. $y = f(x)$, the derivative function may be written as y' or $\frac{dy}{dx}$ or $f'(x)$.

It is important to understand that d/dx is a symbol that means "take the derivative of something with respect to x". For example

$$\frac{d}{dx}\left(x^2\right)$$

means take the derivative of x^2 with respect to x.

Operational Rules

Here are some rules for these are essential and underlie much of what you will use in your course. In this section we present the rules and give some examples.

- If $g(x) = kf(x)$, where k is a constant then

$$g'(x) = kf'(x)$$

or

$$\frac{dg}{dx} = k\frac{df}{dx}. \tag{1}$$

- If $f(x) = k$, where k is a constant then

$$f'(x) = 0 \tag{2}$$

- If $f(x) = g(x) + h(x)$ then

$$f'(x) = g'(x) + h'(x)$$

or

$$\frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx} \tag{3}$$

- Derivative of a power of x. If $y = x^n$, where n is a real number, then

$$\frac{dy}{dx} = nx^{n-1}. \tag{4}$$

Examples

1. If $y = x^7$, find dy/dx.
   Solution:
   Using operational rule (4) we have

$$\begin{aligned} \frac{dy}{dx} &= 7x^{7-1} \\ &= 7x^6. \end{aligned}$$

2. If $y = x^{2000}$, find dy/dx.
   Solution:
   Using operational rule (4) we have

$$\begin{aligned} \frac{dy}{dx} &= 2020x^{2020-1} \\ &= 2020x^{2019}. \end{aligned}$$

3. If $y = \sqrt{x}$, find dy/dx.
   Solution:
   First write y in index form:

$$\begin{aligned} y &= \sqrt{x} \\ &= x^{1/2}. \end{aligned}$$

   Now apply operational rule (4)

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

4. If $y = 1/x^2$, find dy/dx.
Solution:
First write y in index form:

$$\begin{aligned} y &= 1/x^2 \\ &= x^{-2}. \end{aligned}$$

Now apply operational rule (4)

$$\begin{aligned} \frac{dy}{dx} &= -2x^{-2-1} \\ &= -2x^{-3} \\ &= -\frac{2}{x^3}. \end{aligned}$$

5. If $y = x^3 + 7x^6$, find dy/dx.
Solution:
Using operational rule (3),

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(7x^6\right).$$

Now applying operational rule (4) to the first terms on the right we have

$$\frac{dy}{dx} = 3x^2 + \frac{d}{dx}\left(7x^6\right).$$

The second term to the right is dealt with using operational rule (1) and so

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 + 7\frac{d}{dx}\left(x^6\right) \\ &= 3x^2 + 42x^5. \end{aligned}$$

6. If $y = \frac{4}{x}$ +10, find dy/dx.
Solution:
Convert to index form

$$y = 4x^{-1} + 10.$$

Then using operational rules (4)and (3)

$$\frac{dy}{dx} = -4x^{-2} + \frac{d}{dx}(10).$$

Applying operational rule (2)to the last term on the right gives:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{4}{x^2} + 0 \\ &= -\frac{4}{x^2}. \end{aligned}$$

7. If $f(x) = \frac{x^4-1}{\sqrt{x}}$ find $f'(x)$.
Solution:
First convert to index form

$$f(x) = \frac{x^4 - 1}{x^{1/2}},$$

then divide through by $x^{1/2}$ to get

$$\begin{aligned} f(x) &= x^{4-\frac{1}{2}} - x^{-1/2} \\ &= x^{7/2} - x^{-1/2}. \end{aligned}$$

Applying operational rules (3) and (4) we have

$$\begin{aligned} f'(x) &= \frac{7}{2}x^{\frac{7}{2}-1} - \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} \\ &= \frac{7}{2}x^{\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}. \end{aligned}$$

Derivatives of Some Other Functions

The table below gives derivatives for particular functions. Using the table and the rules above we ca expand the functions that we can differentiate.

Function	Derivative
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \log_e x$	$f'(x) = \frac{1}{x}$
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$f'(x) = \sec^2 x$

Examples

1. If $y = \sin(x) + 3x^2$ then $dy/dx =$cos $(x) + 6x$.

2. If $y = e^x - 5$ then $dy/dx = e^x$.

3. If $y = 10\left(\cos(x) - \frac{1}{2x}\right)$ then

$$\begin{aligned} \frac{dy}{dx} &= 10\left(-\sin(x) - \frac{1}{2}(-1)x^{-2}\right) \\ &= -10\sin(x) + \frac{10}{2}x^{-2} \\ &= -10\sin(x) + \frac{5}{x^2}. \end{aligned}$$

Exercises

1. Differentiate the following

a) $y = x^7$ b) $y = x^{\frac{1}{5}}$ c) $y = x^{-19}$ d) $y = \frac{1}{x^4}$ e) 53

f) $y = \sqrt[4]{x}$ g) $5x^6$ h) $y = 9x^{-5}$ i) $\frac{\sqrt{5}}{x}$ j) $3x^2 + 2x$

Answers:

1. a) $7x^6$ b) $\frac{1}{5x^{\frac{4}{5}}}$ c) $-19x^{-20}$ d) $-\frac{4}{x^5}$ e) 0

f) $\frac{1}{4}x^{-\frac{3}{4}}$ g) $30x^5$ h) $-45x^{-6}$ i) $\frac{-\sqrt{5}}{x^2}$ j) $6x + 2$

2. Find the derivatives of

a) $\sin(x) - \cos(x)$ b) $10 - \log_e(x)$ c) $\tan(x) - \sqrt{x}$ d) $3\cos x - \frac{1}{x^2}$ e) $\frac{e^x}{6} - x^7$.

Answers:

2 a) $\cos x + \sin x$ b) $-\frac{1}{x}$ c) $\sec^2(x) - \frac{1}{2\sqrt{x}}$ d) $-3\sin(x) + \frac{2}{x^3}$ e) $\frac{e^x}{6} - 7x^6$.