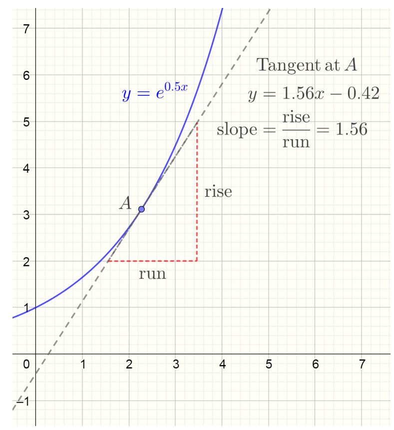
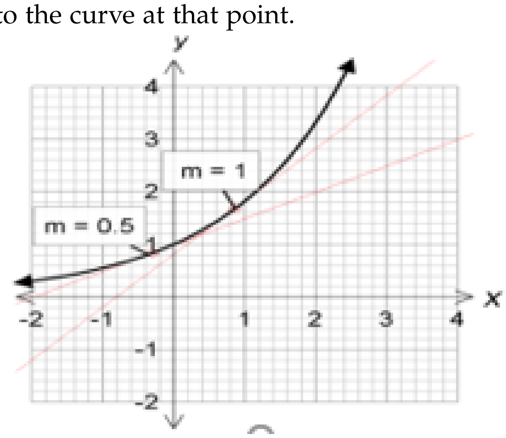
D2 : *Gradients, Tangents and Derivatives*



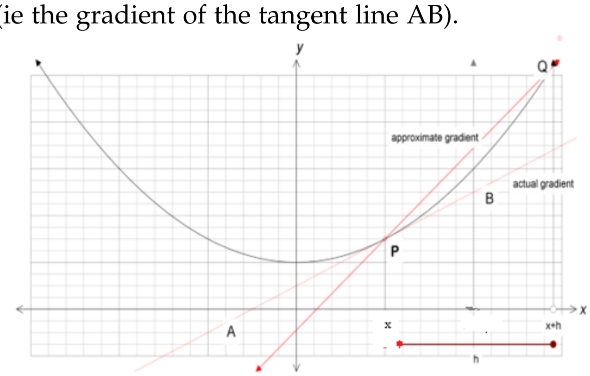
Gradient of a Curve

In this module we are concerned with finding a formula for the slope or gradient of the tangent at any point on a given curve y=$f(x)$. The gradient at a point on a curve is defined as the gradient of the tangent to the curve at that point.



The formula $m = \frac{y_2-y_1}{x_2-x_1}$ may be used to find the gradient of a line when two points on the line, (x_1, y_1) and (x_2, y_2) are known[1].

Consider the gradient of the curve defined by $y = f(x)$ at the point P (ie the gradient of the tangent line AB).



[1] There are two special cases that have to be dealt with: horizontal and vertical lines.

A horizontal line parallel to the x-axis with equation of the form $y = k$ where k is a constant, has a gradient of zero. As a line becomes closer to vertical its gradient gets larger. A vertical line parallel to the y-axis with equation of the form $x = c$ where c is a constant has a gradient which is undefined.

This gradient cannot be calculated as only one point (the point P) on the line is known. But the point P has coordinates $(x, f(x))$and the point Q has coordinates $(x + h, f(x + h))$. The gradient of the line PQ

can be calculated and this can be used to approximate the gradient of AB. The gradient of PQ = $\frac{f(x+h)-f(x)}{h}$. As the value of h decreases (i.e Q becomes closer to the point P), the approximation of the gradient is more accurate. The value of the gradient becomes most accurate as h approaches zero.
The gradient formula for the curve $y = f(x)$ is defined as the derivative function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, h \neq 0.$$

The derivative function$f'(x)$ gives the slope of the tangent to the curve$f(x)$ at any point x.

Example

The derivative of the function $f(x) = \frac{3}{x}$ is $f'(x) = -\frac{3}{x^2}$. Find the slope of the tangent to the curve at $x = 4$.

Solution

At x = 4,

$$\begin{aligned} f'(x) &= -\frac{3}{4^2} \\ &= -\frac{3}{16}. \end{aligned}$$

Hence the slope of the tangent at $x = 4$ is $-\frac{3}{16}$.

Exercises

1. If the derivative function for $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$, find the slope of the tangent to this curve at

a) $x = 2$ b) $x = 0$ c) $x = -9$.

Answer: a) 11 b) -1 c) $x = 242$.

2. If the derivative function of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$ find the gradient of $f(x) = \sin(x)$ at

a) $x = 0$ b) $x = \pi/2$ c) $x = 3.5$.

Answer: a) 1 b) 0 c) -0.94.

3. Determine $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$ and hence find the slope of the tangent to the curve $y = x^2$ at

a) $x = 2$ b) $x = 0$ c) $x = -9$.

Answer: $lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = 2x$.

a) 4 b) $x = 0$ c) -18 .