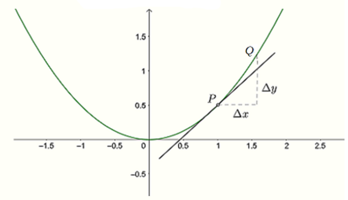
D11:*Small Changes and Approximation*



$$\Delta y \approx \frac{dy}{dx}\Delta x$$

The curve of any differentiable function will appear linear if observed over a sufficiently limited interval. If a is a point within that interval the tangent line through $(a, f(a))$ provides a reasonable approximation of values of $f(x)$ providing x remains near a. A function that is differentiable at a specific point $(a, f(a))$ has a tangent line defined by the equation

$$y = f(a) + f'(a)(x - a)$$

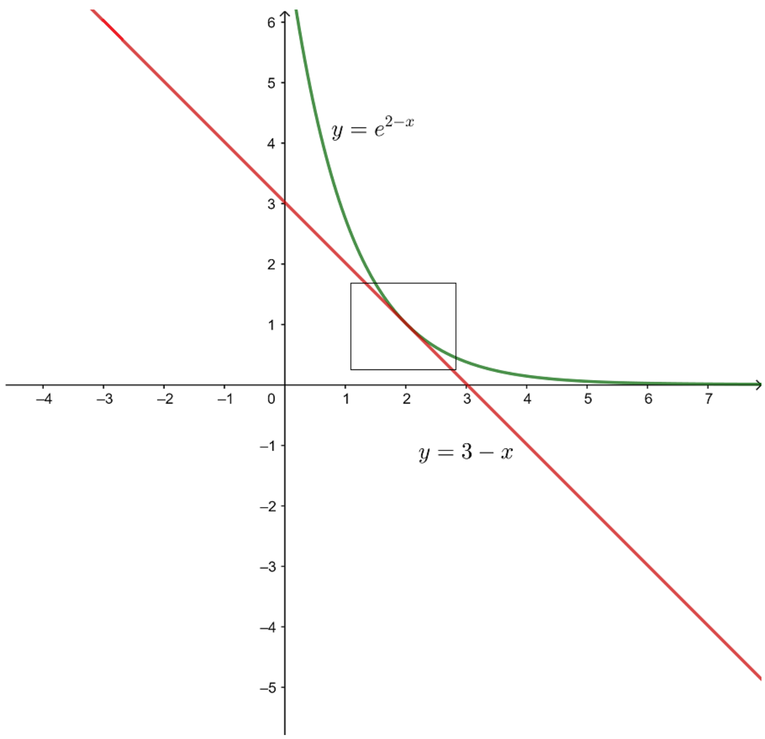
Using the tangent line to approximate a function is called *linear approximation:*

$$L(x) = f(a) + f'(a)(x - a)$$

Example:

Consider the graph of the function $f(x) = e^{2-x}$ near $x = 2$.

Note that $f'(x) = -e^{2-x}$



The tangent line at $x = 2$ is given by:

$$y = f(2) + f'(2)(x - 2)$$
$$y = 1 + (-1)(x - 2)$$
$$y = 3 - x$$

For values of x near 2 the graph of the tangent line is close to the graph of $f(x) = e^{2-x}$ and the tangent line can be used to approximate $f(x)$.

If $x = 1.9$, $f(1.9) = e^{(2-1.9)} \approx 1.105$. At $x = 1.9$ on the tangent line the value of y is $y = 3 - 1.9 = 1.1$

If $x = 2.05$, $f(2.05) \approx 0.951$. At $x = 2.05$ on the tangent line the value of y is 0.95

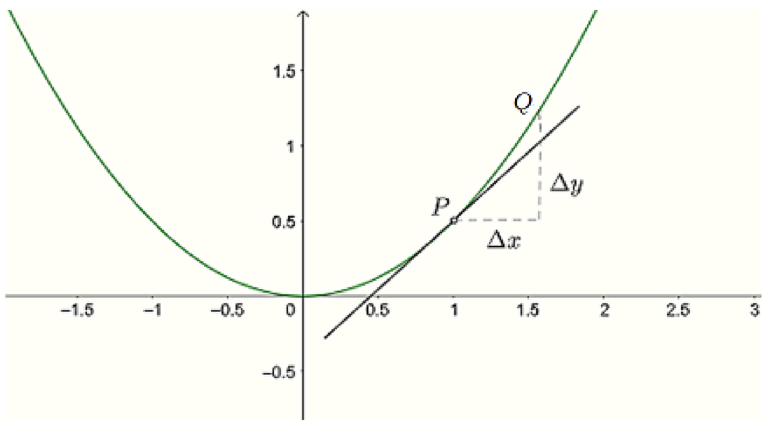
Example:

Use linear approximation of the function $f(x) = \sqrt[3]{x}$ *near* $a = 8$ *to find a value for* $\sqrt[3]{7.98}$

$$f(x) = \sqrt[3]{x} \Rightarrow f'(x) = \frac{x^{-\frac{2}{3}}}{3}$$
$$f(8) = \sqrt[3]{8} = 2 \text{ and } f'(8) = \frac{8^{-\frac{2}{3}}}{3} = \frac{1}{12}$$
$$L(x) = f(a) + f'(a)(x - a)$$
$$L(7.98) = 2 + \frac{1}{12}(7.98 - 8)$$
$$= 1.998$$

Differentials

$\triangle x$ is used to indicate a change in x. $\triangle y$ is used to indicate a change in y. $\triangle x$ and $\triangle y$ are called differentials. Differentials can be used to estimate the amount a function will change as a result of a small change in x.



Consider the graph of the function $y = f(x)$ above and the tangent to the function at the point P. As illustrated in the diagram for values of x close to 1 the slope of the tangent at 1, $f'(1)$, is reasonably approximated by $\frac{\triangle y}{\triangle x}$

$$\text{More generally, as} \triangle x \to 0, \frac{\triangle y}{\triangle x} \to \frac{dy}{dx}$$
$$\text{For small values of } \triangle \text{x: } \triangle y \approx \frac{dy}{dx} \triangle x$$

Examples

1. The side of a square has length 5 cm. How much will the area of the square increase when the side length is increased by 0.01 cm? Let the area of the square be A and let the length of a side be $x\,cm$.

Then $A = x^2$ and $\frac{dA}{dx} = 2x$.

$$\begin{aligned} \triangle A &\approx \frac{dA}{dx} \triangle x \\ &= 2x \triangle x \\ &= 2 \times 5 \times 0.01 \\ &= 0.1 \end{aligned}$$

The increase in area is approximately $0.1\, cm^2$.

2. A 2% error is made in measuring the radius of a sphere. Find the percentage error in the volume?
Let the radius be r and the volume be V then $\Delta r = 0.02r$. Since the volume of the sphere is given by $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$.
First find $\triangle V$:

$$\begin{aligned} \triangle V &\approx \frac{dV}{dr} \triangle r \\ &= 4\pi r^2 \triangle r \\ &= 4\pi r^2 \times 0.02r \\ &= 0.08\pi r^3 \end{aligned}$$

The percentage error in the volume:

$$\begin{aligned} \text{Percentage error in } V &\approx \frac{\Delta V}{V} \times 100\% \\ &= \frac{0.08\pi r^3}{\frac{4}{3}\pi r^3} \times 100\% \\ &= \frac{0.08}{\frac{4}{3}} \times 100\% \\ &= 6\%. \end{aligned}$$

The percentage error in the volume is approximately 6%.

Exercises

1. Use linear approximation of the function $f(x) = \sqrt{x}$ to find the approximate value of $\sqrt{16.1}$
2. Use linear approximation to find an approximate value of $\cos 59°$. (Hint: convert degrees to radians: $1° = \frac{\pi}{180}$ *Radians*.)
3. If the radius of a sphere is increased from $10\, cm$ to $10.1\, cm$ what is the approximate increase in surface area?
4. The height of a cylinder is $10\, cm$ and its radius is $4\, cm$. Find the approximate increase in volume when the radius increases to $4.02\, cm$.

5. An error of 3% is made in measuring the radius of a sphere. Find the percentage error in volume.

Hint: useful formula
Surface area of sphere , $A = 4\pi r^2$
Volume of cylinder , $V = \pi r^2 h$
Volume of sphere , $V = \frac{4}{3}\pi r^3$

Answers

1. 4.0125
2. 0.515
3. 25.13 cm^2
4. 5.03cm^3
5. 9%