IN8 Integration by Parts

$$\int v(x)\frac{du}{dx}dx = v(x)u(x)dx - \int u(x)\frac{dv}{dx}dx$$

abbreviated to

$$\int vu'\,dx = vu - \int uv'\,dx$$

Integration by parts is a technique for integrating the product of two functions. For example:

$$\int x\cos(x)\,dx$$

where the functions are x and $\cos(x)$ or

$$\int x^3\ln(x)\,dx$$

where the functions are x^3 and $\ln(x)$. This module gives some information on using this technique.

Formula

The integration by parts formula is obtained from the product rule for differentiation. For any two differential functions $v(x)$ and $u(x)$ we have, by the product rule,

$$\frac{d}{dx}\left(v(x)\,u(x)\right) = v(x)\frac{du(x)}{dx} + u(x)\frac{dv(x)}{dx}.$$

Integrating both sides with respect to x gives

$$\int \frac{d}{dx}\left(v(x)\,u(x)\right)dx = \int\left(v(x)\frac{du(x)}{dx} + u(x)\frac{dv(x)}{dx}\right)dx, \text{ that is,}$$

$$v(x)\,u(x) = \int v(x)\frac{du(x)}{dx}dx + \int u(x)\frac{dv(x)}{dx}dx.$$

This can be rearranged to give the formula for integration by parts:

$$\int v(x)\frac{du(x)}{dx}dx = v(x)\,u(x) - \int u(x)\frac{dv(x)}{dx}dx.$$

Which is often abbreviated to

$$\int vu'dx = vu - \int uv'dx.$$

The hope is that by choosing the functions v and u' carefully, the integral on the right hand side is easier than the original integral.

Successful use of integration by parts requires a correct choice for v and u'. If the integral on the right is harder, then try swapping v and u'.

To assist you in choosing v and u' choose v to be the function that comes first in the sequence of functions:

1. logarithmic function
2. inverse trig function
3. algebraic function
4. trigonometric function
5. exponential function

u' is then the other function in the integrand.[1].

[1] The integrand is the function that is being integrated. For example the integrand in

$$\int x \sin(x)\, dx$$

is $x\sin(x)$. In this case it is the product of an algebraic function x and a trigonometric function $\sin(x)$ so we would set $v = x$, (because algebraic functions are higher on the list than trigonometric functions), and $u' = \sin(x)$.

Example 1: Find $\int x\cos(x)\, dx$.

Here we have an algebraic function x and a trigonometric function $\cos(x)$. Since algebraic functions are higher on the list than trigonometric functions, we let $v = x$ and so $u' = du/dx = \cos(x)$. Then $v' = dv/dx = 1$ and $u = \sin(x)$. Using the formula we have[2]:

$$\begin{aligned}\int x\cos(x)\, dx &= vu - \int uv' dx \\ &= x\sin(x) - \int (\sin(x)) \cdot 1 dx \\ &= x\sin(x) + \cos(x) + c,\ c \text{ a constant.}\end{aligned}$$

[2] We use

$$\int vu' dx = vu - \int uv' dx.$$

and the fact that $\sin(x) = \sin(x) \times 1$.

Note that if we chose the functions as $v = \cos(x)$ and $u' = x$, then $v' = -\sin(x)$ and $u = \frac{1}{2}x^2$. Substituting in the formula will give:

$$\begin{aligned}\int x\cos(x)\, dx &= vu - \int uv' dx \\ &= -\frac{1}{2}x^2\sin(x) - \int -\frac{1}{2}x^2\sin(x)\, dx.\end{aligned}$$

The integral on the right is harder than the one we began with and we make no progress. If this happens to you, change the selection.

Example 2: Find $\int x^3 \ln(x)\, dx$.

Here we have an algebraic function x^3 and a logarithmic function $\ln(x)$. Since logarithmic functions are higher on the list than algebraic functions, we let $v = \ln(x)$ and so $u' = du/dx = x^3$. Then $v' = dv/dx = 1/x$ and $u = \frac{1}{4}x^4$. Using the formula[3] we have

[3] We use

$$\int vu' dx = vu - \int uv' dx.$$

$$
\begin{aligned}
\int x^3 \ln(x)\, dx &= vu - \int uv' dx \\
&= \ln(x)\left(\frac{1}{4}x^4\right) - \int \frac{1}{4}x^4 \left(\frac{1}{x}\right) dx \\
&= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\int x^3 dx \\
&= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\left(\frac{1}{4}x^4\right) + c, \quad c \text{ a constant} \\
&= \frac{x^4 \ln(x)}{4} - \frac{x^4}{16} + c.
\end{aligned}
$$

Example 3: Find $\int \sin^{-1}(x)\, dx$.

We apparently have only one function $\sin^{-1}(x)$. The trick here is to note that the integrand[4] can always be multiplied by 1 without change. So instead of seeing $\int \sin^{-1}(x)\, dx$. We see $\int \sin^{-1}(x) \times 1 dx$. We can view 1 as the constant value function[5]. The constant value function is algebraic, $\left(1 = x^0\right)$ and below an inverse trigonometric function like $\sin^{-1}(x)$ on the list above. So we set $v = \sin^{-1}(x)$ and $du/dx = 1$, then $dv/dx = 1/\sqrt{1-x^2}$ and $u = x$. Substituting in the formula[6] we get :

[4] The integrand is the function that is integrated. In this case it is $\sin^{-1}(x)$ which is the same as $\sin^{-1}(x) \times 1$.

[5] The constant value function 1 is just a function that gives the value 1 regardless of its argument. That is $1(x) = 1$ for all $x \in \mathbb{R}$.

[6] We use

$$\int vu' dx = vu - \int uv' dx.$$

$$
\begin{aligned}
\int \sin^{-1}(x)\, dx &= vu - \int uv' dx \\
&= x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} dx. \qquad (1)
\end{aligned}
$$

The integral on the right hand side can be evaluated with the substitution $w = 1 - x^2$ then $dw/dx = -2x$ and $x = -\frac{1}{2}\frac{dw}{dx}$, using the substitution rule[7] we get

[7] The substitution rule states

$$\int f(g(x)) \frac{dg}{dx} dx = \int f(w)\, dw$$

where $w = g(x)$.

$$
\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}} dx &= \int -\frac{1}{2}\frac{dw}{dx}\frac{1}{\sqrt{w}} dx \\
&= -\frac{1}{2}\int w^{-1/2} dw \\
&= -\frac{1}{2}(2)\, w^{1/2} + c, \; c \in \mathbb{R} \\
&= -\left(1 - x^2\right)^{1/2} + c \\
&= -\sqrt{1-x^2} + c.
\end{aligned}
$$

Substituting this result in equation (1) we get the final result

$$
\begin{aligned}
\int \sin^{-1}(x)\, dx &= vu - \int uv' dx \\
&= x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} dx. \\
&= x \sin^{-1}(x) + \sqrt{1-x^2} + c.
\end{aligned}
$$

Repeated Use of Integration by Parts

Some integrals require the use of the integration by parts technique more than once. For example, consider $\int x^2 e^x dx$. We have an algebraic function x^2 multiplying an exponential function e^x. According to our guide we set $v(x) = x^2$ and $u'(x) = e^x$. Then $dv/dx = 2x$ and $u(x) = e^x$. Substituting in the integration by parts formula[8] we get:

$$\int x^2 e^x dx = x^2 e^x - \int e^x (2x)\, dx. \quad (2)$$

[8] We use

$$\int vu' dx = vu - \int uv' dx.$$

The integral on the right hand side requires another integration by parts. We set $v(x) = 2x$ and $u'(x) = e^x$. Then $dv/dx = 2$ and $u(x) = e^x$. Substituting in the integration by parts formula gives

$$\begin{aligned} \int e^x (2x)\, dx &= 2xe^x - \int 2e^x dx \\ &= 2xe^x - 2e^x + c, \ c \in \mathbb{R}. \quad (3) \end{aligned}$$

Now substitute this result in equation (2) to get the result

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - \int e^x (2x)\, dx \\ &= x^2 e^x - (2xe^x - 2e^x + c) \text{ using (3)} \\ &= x^2 e^x - 2xe^x + 2e^x - c \\ &= e^x \left(x^2 - 2x + 2\right) - c. \end{aligned}$$

Note that it doesn't matter if we put c or $-c$ as c is just any constant.

Exercises

1. Find:

$a)$ $\int x \sin(x)\, dx$ $b)$ $\int xe^x dx$

$c)$ $\int x \cos(4x)\, dx$ $d)$ $\int x^4 \ln(x)\, dx$

2. Find:

$a)$ $\int x^2 \cos(x)\, dx$ $b)$ $\int x^3 e^{x^2}$ (Hint: let $v = x^2$)

Answers

1. $a)$ $-x\cos(x) + \sin(x) + c$ $b)$ $xe^x - e^x + c$

$c)$ $\frac{1}{4}x\sin(4x) + \frac{1}{16}\cos(4x) + c$ $d)$ $\frac{x^5}{5}\ln(x) - \frac{x^5}{25} + c$

2. $a)$ $x^2\sin(x) + 2x\cos(x) - 2\sin(x) + c$

$b)$ $x^2 e^{x^2} - e^{x^2} + c$