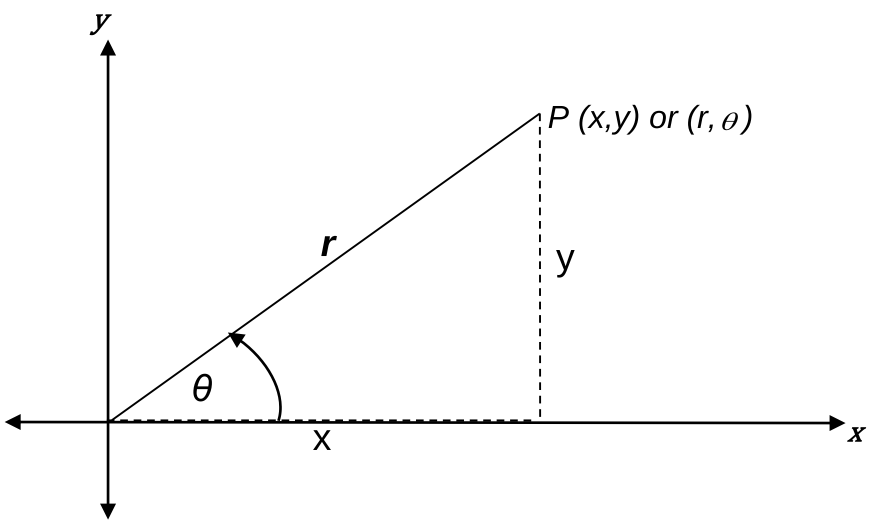
RMIT UNIVERSITY

CN2 POLAR FORM OF A COMPLEX NUMBER

Rectangular and Polar Form

When a complex number is expressed in the form $z = x + yi$ it is said to be in *rectangular form.*

But a point P with Cartesian coordinates (x,y) can also be represented by the polar coordinates (r, θ) where r is the distance of the point P from the origin and θ is the angle that $\overrightarrow{OP}$ makes with the positive x-axis



NB: $x = r\cos\theta$ and $y = r\sin\theta$ and $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$

To express a complex number z in polar form:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + r\sin\theta\, i \\ &= r(\cos\theta + \sin\theta\, i) \end{aligned}$$

which we abbreviate to $z = r\text{cis}\ \theta$

So, the polar form of the complex number z is

$$z = r\ cis\ \theta$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Modulus and argument of z

The *modulus* of z, $|z|$ is the distance of the point z from the origin.

$$\text{mod } z = |z| = |x+yi| = \sqrt{x^2+y^2} = r$$

The *argument* of z, arg z, is the angle measured from the positive direction of the x-axis to $\overrightarrow{OP}$

If arg z = θ then $\sin\theta = \dfrac{y}{|z|}$ and $\cos\theta = \dfrac{x}{|z|}$ and $\tan\theta = \dfrac{y}{x}$

An infinite number of arguments of z exist, for example, if z = i then arg z = $\dfrac{\pi}{2} + 2\pi n, n \in Z$.

argument and Argument of z

We define the Argument of z: **Arg z =** θ**, where** $-\pi \le \theta \le \pi$

So while the argument (with a small "a") of z has many values, the Argument (with a capital "A") of z has only one value.

Examples

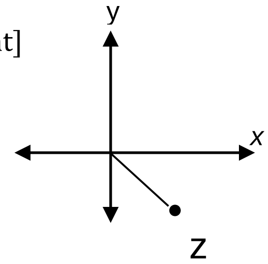
1. *Express in polar form z = 1 – i*

x = 1, y = -1 [NB: z is in the 4th quadrant]

$r = |z| = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \dfrac{y}{x} = \dfrac{-1}{1} = -1$

$\theta = \tan^{-1}(-1) = \dfrac{-\pi}{4}$ [since z is in the 4th quadrant]



$\therefore z = r\,cis\theta$

$= \sqrt{2}\text{ cis}\left(\dfrac{-\pi}{4}\right)$

2. *Express 2 cis* $\left(\dfrac{4\pi}{3}\right)$ *in the form x + yi*

$2\text{ cis}\left(\dfrac{4\pi}{3}\right) = 2\left[\cos\left(\dfrac{4\pi}{3}\right) + \sin\left(\dfrac{4\pi}{3}\right)i\right]$

$= 2 \times \left(-\dfrac{1}{2}\right) + 2 \times \left(-\dfrac{\sqrt{3}}{2}\right)i$

$= -1 - \sqrt{3}\,i$

See Exercise 1

Operations on Complex Numbers in Polar Form

Addition and Subtraction

Complex numbers in polar form are best converted to the form x + yi before addition or subtraction

Multiplication and Division

If $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$ then it can be shown using trigonometric identities that

$$z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Examples:

1. If $z_1 = 2cis\frac{\pi}{4}$ and $z_2 = -3cis\frac{5\pi}{6}$ find $z_1 z_2$ in polar form, $-\pi \le \theta \le \pi$

$$z_1 z_2 = 2\text{cis}\frac{\pi}{4} \times \left(-3\text{cis}\frac{5\pi}{6}\right)$$

$$= -6\text{cis}\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)$$

$$= -6\text{cis}\left(\frac{13\pi}{12}\right)$$

$$= -6\text{cis}\left(\frac{-11\pi}{12}\right) \quad \text{since } -\pi \le \theta \le \pi$$

2. If u = 1 + 3i and v = 2 – i find $\frac{u}{v}$ in polar form with $-\pi \le \theta \le \pi$

There are two possible approaches to this problem

u = 1 + 3i ie. x = 1, y = 3 **OR**

$$r = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\theta = \tan^{-1}\left(\frac{3}{1}\right) = 1.25 \textit{ radians}$$

$$\therefore \; u = \sqrt{10}\,\text{cis}1.25,$$

v = 2 – i ie x = 2, y = -1

$$r = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = -0.46$$

$$\therefore \; v = \sqrt{5}\,\text{cis}0.46$$

$$\text{Then } \frac{u}{v} = \frac{\sqrt{10}cis1.25}{\sqrt{5}cis(-0.46)}$$

$$= \frac{\sqrt{10}}{\sqrt{5}}\text{cis}(1.25 + 0.46)$$

$$= \sqrt{2}\text{ cis } 1.71$$

$$\frac{u}{v} = \frac{1+3i}{2-i}$$

$$= \frac{1+3i}{2-i} \times \frac{2+i}{2+i}$$

$$= \frac{-1+7i}{4+1}$$

$$= \frac{-1+7i}{5}$$

$$= -\frac{1}{5} + \frac{7}{5}i$$

$$\therefore \; x = -\frac{1}{5}, \quad y = \frac{7}{5}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-0.2)^2 + 1.4^2} = \sqrt{2}$$

$$\tan\theta = \frac{1.4}{-0.2} = -7, \; \theta = 1.71 \textit{ radians}$$

$$\therefore \frac{u}{v} = \sqrt{2}\text{ cis}(1.71)$$

See Exercise 2

Exercise 1

1. Find the polar form (in radians) of the following complex numbers:

(a) z = -1 + i (b) $z = -\sqrt{3} + i$

(c) z = - 3i (d) z = -2 – 4i

2. Express each of the following complex numbers in rectangular form

(a) $3\text{cis}\frac{\pi}{4}$ (b) $\sqrt{7}cis\pi$

(c) $8\text{cis}\frac{\pi}{2}$ (d) 10cis0.41

3. If z = 2 + i and w = 1 – 4i find each of the following in polar form using radians where appropriate:

(a) $|z|$ (b) $|w|$ (c) Arg z (d) $|\overline{w}|$

(e) Arg(zw) (f) zw

Exercise 2

1. Simplify

(a) $4cis\frac{\pi}{3} \times 3cis\frac{\pi}{4}$ (b) $\dfrac{3cis\frac{5\pi}{6}}{12cis\frac{\pi}{6}}$

2. If $u = 6\text{cis}\frac{3\pi}{4}$ and $v = 4\text{cis}\left(-\frac{\pi}{4}\right)$ express $\frac{u}{v}$ in polar form

3. If $z = 1 - \sqrt{3}i$, find $\overline{z}$ and express both z and $\overline{z}$ in polar form using radians.

Answers
Exercise 1

1. (a) $\sqrt{2}cis\frac{3\pi}{4}$ (b) $2cis\frac{5\pi}{6}$ (c) $3cis\frac{-\pi}{2}$ (d) $\sqrt{20}cis(-2.03)$

2. (a) $\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$ (b) $-\sqrt{7}$ (c) 8i (d) 9.2 + 4i

3. (a) $\sqrt{5}$ (b) $\sqrt{17}$ (c) 0.46 (d) $\sqrt{17}$
(e) -0.86 (f) 9.22cis(-0.86)

Exercise 2

1. (a) $12\text{cis}\frac{7\pi}{12}$ (b) $\frac{1}{4}\text{cis}\frac{2\pi}{3}$

2. (a) $\frac{3}{2}cis\pi$

3. $z = 2cis\left(-\frac{\pi}{3}\right)$ $\overline{z} = 2cis\left(\frac{\pi}{3}\right)$