RMIT UNIVERSITY

CN1 COMPLEX NUMBERS: INTRODUCTION

Real and complex numbers

Equations such as $x + 1 = 7$, $3x = 10$ and $x^2 - 7 = 0$ can all be solved within the real number system. But there is no real number which satisfies $x^2 + 1 = 0$.

To obtain solutions to this and other similar equations the ***complex numbers*** were developed.

The ***imaginary*** number i is defined such that $i^2 = -1$

That is $i = \sqrt{-1}$ and $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$ etc

A number z of the form $z = x + yi$ where x and y are real numbers is called a ***complex number***

x is called the ***real part*** of z, denoted by ***Re z***, and

y is called the ***imaginary part*** of z, denoted by ***Im z***

Examples

1. If $z = 5 - 3i$ then $\text{Re } z = 5$ and $\text{Im } z = -3$
2. If $z = \sqrt{3}\, i$ then $\text{Re } z = 0$ and $\text{Im } z = \sqrt{3}$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal

$$a + bi = c + di$$
if and only if
$$a = c \text{ and } b = d$$

Example

If $z_1 = x - \frac{i}{3}$, $z_2 = \sqrt{2} + yi$ *and* $z_1 = z_2$ *find the values of x and y.*

$\text{Re } z_1 = \text{Re } z_2 \quad \Rightarrow \quad x = \sqrt{2}$

and $\text{Im } z_1 = \text{Im } z_2 \quad \Rightarrow \quad y = -\frac{1}{3}$

$\therefore\ x = \sqrt{2}$ and $y = -\frac{1}{3}$

Addition and Subtraction of Complex Numbers

To add or subtract complex numbers we add or subtract the real and imaginary parts separately:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

Examples

1. $(2 + 3i) + (4 - i) = (2 + 4) + (3 - 1)i$

$= 6 + 2i$

2. *If* $z_1 = 1 - i$ *and* $z_2 = 3 - 5i$ *find* $z_1 - z_2$

$z_1 - z_2 = (1 - i) - (3 - 5i)$

$= (1 - 3) + (-1 - (-5))i$

$= -2 + 4i$

See ***Exercise 1***

Multiplication of Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then

$k z_1 = k(a + bi)$

$= ka + kbi$

and

$z_1 z_2 = (a + bi)(c + di)$

$= ac + adi + bci + bdi^2$

$= (ac - bd) + (ad + bc)i \qquad [\text{ since } i^2 = -1]$

Examples

1. *Expand and simplify $i(3 + 4i)$*

$$i(3 + 4i) = 3i + 4i^2$$
$$= -4 + 3i$$

2.. *If* $z_1 = 1 - i$ *and* $z_2 = 3 - 5i$ *find* $z_1 z_2$

$$z_1 z_2 = (1 - i)(3 - 5i)$$
$$= 3 - 3i - 5i + 5i^2$$
$$= 3 - 8i - 5$$
$$= -2 - 8i$$

See ***Exercise 2***

Complex Conjugates

A pair of complex numbers of the form a + bi and a – bi are called ***complex conjugates.***

If $z = x + yi$ then the conjugate of z is denoted by $\overline{z} = x - yi$

Eg: $2 + 3i$ and $2 - 3i$ are a conjugate pair

$1 - i$ and $1 + i$ are a conjugate pair

$-4i$ and $4i$ are a conjugate pair

The product of a conjugate pair of complex numbers is a ***real*** number

Since $z\overline{z} = (x + yi)(x - yi) = x^2 + y^2$

Some properties of conjugates:

If z_1 and z_2 represent two conjugate numbers then:

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (ii) $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$ (iii) $\overline{\overline{z}} = z$

Examples

If $z = 2 - i$ and $w = -3 + 4i$ find: **1.** $\overline{z}$ **2.** $\overline{z} - \overline{w}$ **3.** $\overline{z + w}$

1. $\overline{z} = 2 + i$

2. $\overline{z} - \overline{w} = 2 + i - (-3 - 4i)$
$$= 2 + 3 + i + 4i$$
$$= 5 + 5i$$

3. $\overline{z+w} = \overline{2-i+(-3+4i)}$

$= \overline{-1+3i}$

$= -1 - 3i$

See ***Exercise 3***

Division of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $\frac{z_1}{z_2} = \frac{a+bi}{c+di}$.

To express $\frac{z_1}{z_2}$ in the form x + yi we make use of the conjugate to '*realize*' the denominator.

Examples

1. *Express* $\frac{2-i}{1+3i}$ *in the form x + yi*

$$\frac{2-i}{1+3i} = \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i}$$

$$= \frac{2-i-6i-3}{1+9}$$

$$= \frac{-1-7i}{10}$$

$$= -\frac{1}{10} - \frac{7}{10}i$$

2. $\frac{i}{1-4i} + \frac{2}{3+i} = \frac{i}{1-4i} \cdot \frac{1+4i}{1+4i} + \frac{2}{3+i} \cdot \frac{3-i}{3-i}$ [to rationalize denominators]

$$= \frac{i-4}{1+16} + \frac{6-2i}{9+1}$$

$$= \frac{i-4}{17} + \frac{6-2i}{10}$$

$$= \frac{i-4}{17} \cdot \frac{10}{10} + \frac{6-2i}{10} \cdot \frac{17}{17}$$ [170 is a common denominator]

$$= \frac{10i-40}{170} + \frac{102-34i}{170}$$

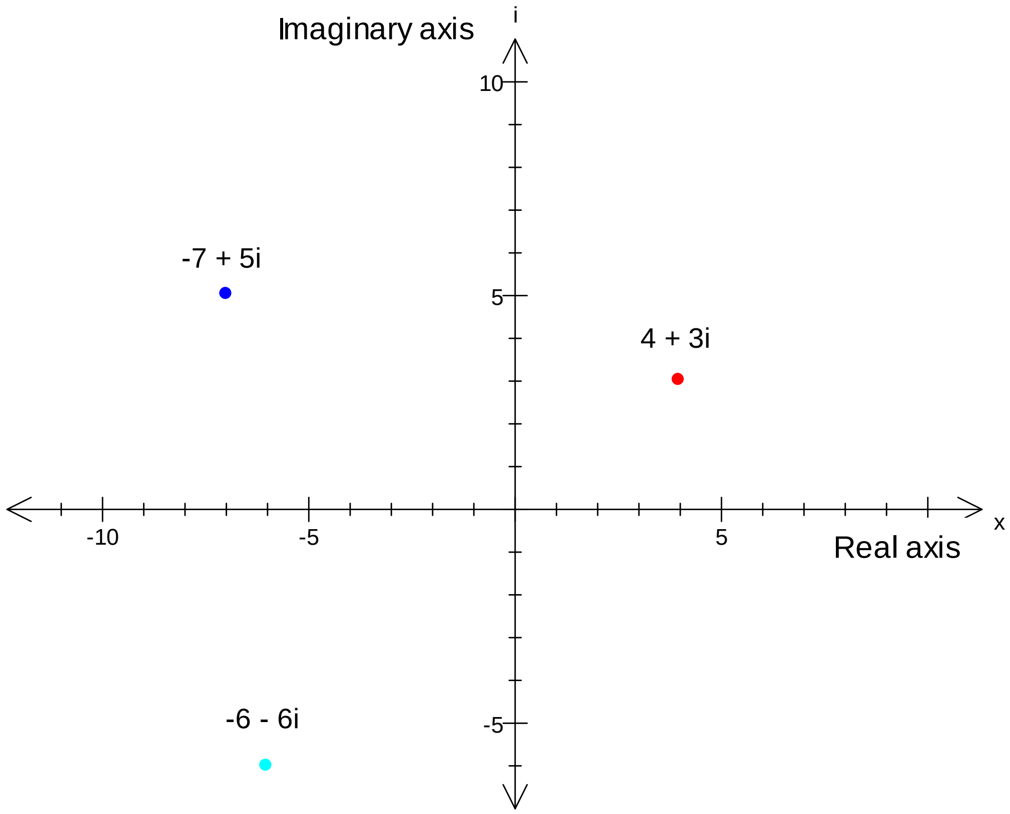
$$= \frac{10i-40+102-34i}{170}$$

$$= \frac{62-24i}{170}$$

$$= \frac{2(31-12i)}{170} = \frac{31-12i}{85}$$

See ***Exercise 4***

An ***Argand Diagram*** is a geometrical representation of the set of complex numbers. The complex number $z = x + yi$ can plotted as a point represented by the ordered pair (x,y) on the complex number plane:



See ***Exercise 5***

Exercises

Exercise 1

1. Express the following in terms of i in simplest surd form

   (a) $\sqrt{-9}$ (b) $\sqrt{-2}$ (c) $\sqrt{-5} \times \sqrt{3}$ (d) $\sqrt{-5} \times \sqrt{10}$ (e) $\sqrt{-6} \times \sqrt{12}$

2. Evaluate

   (a) i^4 (b) i^9 (c) $i^7 - i^{11}$ (d) $i^5 + i^6 - i^7$ (e) $2i - i^6 + 2i^7$

3. State the value of *Re z* and *Im z* for these complex numbers:

   (a) 2 + 7i (b) 10 – i (c) π + 3i (d) $\frac{i}{6}$ (e) -8

4. Find the values of x and y

   (a) x + yi = 4 + 9i (b) x + yi = 3 – i

   (c) x + yi = 23 (d) x + yi = $-\sqrt{2}$ i

   (e) x + i = -5 + yi

Exercise 2

1. Expand and simplify

   (a) i(3 - 2i) (b) 2i³(1 – 5i) (c) (8 – 3i)(2 – 5i) (d) (4 – 3i)² (e) (3 + 2i)(3 – 2i)

2. If z_1 = -1 + 3i and z_2 = 2 - i find each of the following

   (a) $z_1 z_2$ (b) $2z_1 - z_2$ (c) $(z_1 - z_2)^2$

3. Find the value of x and y if (x + yi)(2 – 3i) = -13i

Exercise 3

1. Find the conjugate of each of the following complex numbers:

   (a) 4 + 9i (b) -3-15i (c) $\sqrt{3}$ - 4i

2. Find the conjugate of (2 – i)(4 + 7i)

3. If z = 2 - i and w = 1 + 2i express the following in the form x + yi:

   (a) $\bar{z}$ (b) $\overline{z+w}$ (c) $\bar{z} + \bar{w}$ (d) $\overline{zw}$ (e) $\overline{\bar{z}-\bar{w}}$

Exercise 4

1. Express the following in the form x + yi

   (a) $\dfrac{4-9i}{3}$ (b) $\dfrac{1}{3-i}$ (c) $\dfrac{5+i}{2-7i}$

2. Simplify $\dfrac{2}{1-i}+\dfrac{3+i}{i}$

3. If w = -1 + 6i express $\dfrac{w+1}{w-i}$ in the form x + yi

Exercise 5

If z = 2 – 3i and w = 1 + 4i, illustrate on an Argand diagram

1. z 2. w 3. z + w 4. $\overline{z+w}$ 5. 2z - w

Answers

Exercise 1

1. (a) 3i (b) $\sqrt{2}i$ (c) $\sqrt{15}i$ (d) $5\sqrt{2}i$ (e) $6\sqrt{2}i$

2. (a) 1 (b) i (c) 0 (d) 2i -1 (e) 1

3. (a) $Re\,z = 2$ $Im\,z = 7$ (b) $Re\,z = 10$ $Im\,z =$ -1 (c) $Re\,z = \pi$ $Im\,z = 3$ (d) $Re\,z = 0$ $Im\,z = \frac{1}{6}$ (e) $Re\,z =$ -8 $Im\,z = 0$

4. (a) x = 4, y = 9 (b) x = 3, y = -1 (c) x = 23, y = 0 (d) x = 0, y = $-\sqrt{2}$ (e) x = -5, y = 1

Exercise 2

1. (a) 2 + 3i (b) -10 - 2i (c) 1 – 46i (d) 7 – 24i (e) 13

2. (a) 1 + 7i (b) -4 + 7i (c) -7 – 24i **3.** x = 3, y = -2

Exercise 3

1 (a) 4 – 9i (b) -3 + 15i (c) $\sqrt{3}$ + 4i

2. 15 – 10i 3. (a) 2 + i (b) 3 – i (c) 3 – i (d) 4 – 3i (e) 1 – 3i

Exercise 4

1. (a) $\frac{4}{3}$ - 3i (b) $\frac{3}{10}+\frac{1}{10}i$ (c) $\frac{3}{53}+\frac{37}{53}i$ **2.** 2 – 2i **3.** $\frac{15}{13}-\frac{3}{13}i$

Exercise 5